DIRECTORATE OF DISTANCE EDUCATION

UNIVERSITY OF NORTH BENGAL

MASTERS OF SCIENCE-MATHEMATICS

SEMESTER –II

COMPLEX ANALYSIS-II

DEMATH-2 ELEC-5

BLOCK-2

UNIVERSITY OF NORTH BENGAL

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FOREWORD

The Self-Learning Material (SLM) is written with the aim of providing simple and organized study content to all the learners. The SLMs are prepared on the framework of being mutually cohesive, internally consistent and structured as per the university's syllabi. It is a humble attempt to give glimpses of the various approaches and dimensions to the topic of study and to kindle the learner's interest to the subject

We have tried to put together information from various sources into this book that has been written in an engaging style with interesting and relevant examples. It introduces you to the insights of subject concepts and theories and presents them in a way that is easy to understand and comprehend.

We always believe in continuous improvement and would periodically update the content in the very interest of the learners. It may be added that despite enormous efforts and coordination, there is every possibility for some omission or inadequacy in few areas or topics, which would definitely be rectified in future.

We hope you enjoy learning from this book and the experience truly enrich your learning and help you to advance in your career and future endeavours.

COMPLEX ANALYSIS II

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Unit 2 Analytic And Harmonic Function

Unit 3 Application of Harmonic Function

Unit 4 The Dirichlet Problem for the Unit Disk and Fourier Series

Unit 5 Geometric Series and Convergence

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BLOCK-2

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Unit 10 Factorization in Integral Function

Unit 11 Cauchy's Integral Function

Unit 12 Complex Exponents and Zeros

Unit 13 Genus and Laguerre's Theorem

Unit 14 Borel's and Picard's Theorem

BLOCK-2 COMPLEX ANALYSIS II

Introduction To Block-Ii

Unit 8 Comparison Test For Convergence: Deals with comparison test and definition of series

Unit 9 Absolute and Uniform Convergence: Deals with absolute convergence and uniform convergence series

Unit 10 Factorization in Integral Function: Deals with factorization of integral function and Weierstrass Primary Factors

Unit 11 Cauchy's Integral Function: Deals with Cauchy' integral function and Sokhotskii formula, Fundamental theorem of integration

Unit 12 Complex Exponents and Zeros: Deals with complex exponents, singularities, zeros and poles

Unit 13 Genus and Laguerre's Theorem: Deals with Genus and Laguerre's Theorem with its examples and properties. Also deals with Hadamard's Theorem

Unit 14 Borel's and Picard's Theorem: Deals with Borel's and Picard's Theorem with its statement and proof with examples

UNIT 8: COMPARISON TEST FOR CONVERGENCE

STRUCTURE

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8.0 OBJECTIVE

- We study Comparison test for convergence of series
- We study Definition of series

- We study Geometric series
- We study p-series and telescoping series
- We study Divergent test and its examples

8.1 INTRODUCTION

In mathematics, the **comparison test**, sometimes called the **direct comparison test** to distinguish it from similar related tests (especially the limit comparison test), provides a way of deducing the convergence or divergence of an infinite series or an improper integral. In both cases, the test works by comparing the given series or integral to one whose convergence properties are known.

We have seen that the integral test allows us to determine the convergence or divergence of a series by comparing it to a related improper integral[i]al. In this section, we show how to use comparison tests to determine the convergence or divergence of a series by comparing it to a series whose convergence or divergence is known. Typically these tests are used to determine convergence of series that are similar to geometric series or p-series.

Let Σa_k and Σb_k be a series with positive terms and suppose $a_1 \le b_1$, $a_2 \le b_2$, ...

1. If the bigger series converges, then the smaller series also converges.

2. If the smaller series diverges, then the bigger series also diverges.

8.2 THE DEFINITION OF A SERIES

Let

$$a_1, a_2, a_3, \cdots$$

Be a sequence. We call the sum

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

An *infinite series* (or just a *series*) and denote it as

$$\sum_{n=1}^{\infty} a_n \quad .$$

We define a second sequence, *s*[*n*], called the *partial sums*, by

$$s_1 = a_1,$$

$$s_2 = a_1 + a_2,$$

$$s_3 = a_1 + a_2 + a_3,$$

Or, in general,

$$s_n = \sum_{i=1}^{n} a_i \quad .$$

We then define convergence as follows:

8.2.1 Definition of Convergence, or the Limit of a Series

Given a series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

let *s*[*n*] denote its *n*th partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n \quad .$$

If the sequence *s*[*n*] has a limit, that is, if there is some *s* such that for all $\epsilon > 0$ there exists some *N* > 0 such that |*s*[*n*] - *s*| < ϵ, then the series is called *convergent*, and we say the series *converges*. We write

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = s$$

or

Notes

$$\sum_{n=1}^{\infty} a_n = s$$

The number *s* is called the *sum of the series*. If the series does not converge, the series is called *divergent*, and we say the series *diverges*.

8.2 LIST OF COMMON SERIES

8.2.1 Geometric Series

A series is called geometric if each term in the series is obtained from the preceding one by multiplying it by a common ratio. For example, the series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

Is geometric, since each term is obtained by multiplying the preceding term by 1/2. In general, a geometric series is of the form

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \text{ for } a \neq 0$$

Geometric series are useful because of the following result:

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

Is convergent if $|r| < 1$, and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ for } |r| < 1$$

Otherwise, the geometric series is divergent.

So, for our example above, a=1, and r=1/2, and the sum of the series is

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} 1 \cdot \left(\frac{1}{2}\right)^{n-1} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

8.2.2 p-Series

A series such as

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Is called a *p*-series. In general, a *p*-series follows the following form:

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots$$

P-series are useful because of the following theorem:

The *p*-series

$$\sum_{n=1}^{\infty}\frac{1}{n^p}$$

Is convergent if $p > 1$ and divergent otherwise.

Unfortunately, there is no simple theorem to give us the sum of a *p*-series. For instance, the sum of the example series is

$$1+\frac{1}{4}+\frac{1}{9}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

If p=1, we call the resulting series the *harmonic series*:

$$1+\frac{1}{2}+\frac{1}{3}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n}$$

By the above theorem, the harmonic series does not converge.

8.2.3 Telescoping Series

A telescoping series does not have a set form, like the geometric and *p*-series do. A telescoping series is any series where nearly every term cancels with a preceding or following term. For instance, the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

Notes

Is telescoping. Look at the partial sums:

$$\sum_{i=1}^{n}\left(\frac{1}{i}-\frac{1}{i+1}\right)=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=1-\frac{1}{n+1}$$

Because of cancellation of adjacent terms. So, the sum of the series, which is the limit of the partial sums, is 1.

You do have to be careful; not every telescoping series converges. Look at the following series:

$$\sum_{n=1}^{\infty} n-(n+1)$$

You might at first think that all of the terms will cancel, and you will be left with just 1 as the sum... But take a look at the partial sums:

$$\sum_{i=1}^{n} i-(i+1)=(1-2)+(2-3)+\cdots+(n-(n+1))=$$

$$1-(n+1)=-n.$$

This sequence does not converge, so the sum does not converge. This can be more easily seen if you simplify the expression for the term. You find that

$$\sum_{n=1}^{\infty} n-(n+1)=\sum_{n=1}^{\infty}-1$$

And any infinite sum with a constant term diverges.

Check in Progress-I

Note: Please give solution of questions in space give below:

Q. 1 Give definition of Series.

Solution:

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Q. 2 give the definition of convergence.

Solution:

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8.3 LIST OF SERIES TESTS

The series of interest will always by symbolized as the sum, as *n* goes from 1 to infinity, of *a*[*n*]. In addition, any auxiliary sequence will be symbolized as the sum, as *n* goes from 1 to infinity, of *b*[*n*]. Or, symbolically,

$$\sum_{n=1}^{\infty} a_n \text{ And } \sum_{n=1}^{\infty} b_n .$$

8.3.1 The Common Series Tests

8.3.1.1 The Divergence Test

If the limit of *a*[*n*] is not zero, or does not exist, then the sum diverges.

$$\lim_{n\to\infty} a_n \text{ does not exist, or}$$

$$\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$

For instance, the sum

$$\sum_{n=1}^{\infty} \frac{n+1}{n}$$

Doesn't converge, since the limit as *n* goes to infinity of (*n*+1)/*n* is 1. Note that the implication only goes one way; if the limit *is* zero, you still may not get convergence. For instance, the terms of

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Have a limit of zero, but the sum does *not* converge.

8.3.1.2 The Integral Test

If you can define *f* so that it is a continuous, positive, decreasing function from 1 to infinity (including 1) such that *a*[*n*]=*f(n)*, then the sum will converge if and only if the integral off from 1 to infinity converges.

$$f \text{ continuous, positive, decreasing on } [1, \infty)$$

$$\text{such that } a_n = f(n)$$

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_1^{\infty} f(x)\, dx \text{ converges}$$

For example, look at the sum

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2}.$$

Does it converge? Well, define *f*(*x*) as follows

$$f(x) = \frac{x+1}{x^2}$$

And see if the integral converges.

$$\int_1^{\infty} \frac{x+1}{x^2}\, dx = \int_1^{\infty} (x+1)\, x^{-2}\, dx = \int_1^{\infty} \left(x^{-1} + x^{-2}\right)\, dx =$$

$$\ln x - x^{-1}\Big|_1^{\infty} = \lim_{u \to \infty} \ln u - u^{-1} - \ln 1 + 1^{-1} =$$

$$\lim_{u\to\infty} \ln u - \lim_{u\to\infty} \frac{1}{u} - 0 + 1 = \lim_{u\to\infty} \ln u - 0 - 0 + 1 = \infty$$

The integral does not converge, so the sum does not converge either.

Remember, though, that the value of the integral is not the same as the sum of the series, at least in general. For instance,

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1} = \frac{1}{1-\frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

But

$$\int_1^{\infty} \left(\frac{1}{3}\right)^{x-1} dx = \int_1^{\infty} \exp\left((x-1)\ln\frac{1}{3}\right) dx =$$

$$\left.\frac{\exp\left((x-1)\ln\frac{1}{3}\right)}{\ln\frac{1}{3}}\right|_1^{\infty} = \left.\frac{\exp\left(-(x-1)\ln 3\right)}{-\ln 3}\right|_1^{\infty} =$$

$$0 - \frac{\exp(-0\cdot\ln 3)}{-\ln 3} = \frac{1}{\ln 3} \neq \frac{3}{2}.$$

8.3.1.3 The Comparison Test

Let *b*[*n*] be a second series. Require that all *a*[*n*] and *b*[*n*] are positive. If *b*[*n*] converges, and *a*[*n*] <=*b*[*n*] for all *n*, then *a*[*n*] also converges. If the sum of *b*[*n*] diverges, and *a*[*n*]>=*b*[*n*] for all *n*, then the sum of *a*[*n*] also diverges.

$$a_n, b_n > 0 \text{ for all } n$$

$$\sum_{n=1}^{\infty} b_n \text{ converges}, a_n \leq b_n \text{ for all } n \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

$$\sum_{n=1}^{\infty} b_n \text{ diverges}, a_n \geq b_n \text{ for all } n \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$

The idea with this test is that if each term of one series is smaller than another, then the sum of that series must be smaller. So, if every term of a series is smaller than the corresponding term of a converging series, the smaller series must also converge. And if a smaller series diverges, the larger one must also diverge.

As an example, consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n+1}.$$

Compare that with a second series as follows:

$$\frac{1}{n+1} > \frac{1}{2n} \text{ (Since } n+1<2n \text{ for } n>=1\text{)} = \frac{1}{2} \cdot \frac{1}{n}.$$

$$\sum_{n=1}^{\infty} \frac{1}{2} \cdot \frac{1}{n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}.$$

Since this new, smaller sum diverges (it is a harmonic series), the original sum also diverges.

For another example, look at

$$\sum_{n=1}^{\infty} \frac{n-1}{n^3}.$$

Compare that with a second series also:

$$\frac{n-1}{n^3} = \frac{n}{n^3} - \frac{1}{n^3} < \frac{1}{n^2}.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Converges (since it is a *p*-series with *p* greater than one), so the first sum also converges.

8.3.1.4 The Limit Comparison Test

Let *b*[*n*] be a second series. Require that all *a*[*n*] and *b*[*n*] are positive.

- If the limit of *a*[*n*]/*b*[*n*] is positive, then the sum of *a*[*n*] converges if and only if the sum of *b*[*n*] converges.

- If the limit of *a*[*n*]/*b*[*n*] is zero, and the sum of *b*[*n*] converges, then the sum of *a*[*n*] also converges.
- If the limit of *a*[*n*]/*b*[*n*] is infinite, and the sum of *b*[*n*] diverges, then the sum of *a*[*n*] also diverges.

$$a_n, b_n > 0 \text{ for all } n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} > 0 \Rightarrow \left(\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} b_n \text{ converges} \right)$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0 \text{ and } \sum_{n=1}^{\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty \text{ and } \sum_{n=1}^{\infty} b_n \text{ diverges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$

Here we are comparing how fast the terms grow. If the limit is positive, then the terms are growing at the same rate, so both series converge or diverge together. If the limit is zero, then the bottom terms are growing more quickly than the top terms. Thus, if the bottom series converges, the top series, which is growing more slowly, must also converge. If the limit is infinite, then the bottom series is growing more slowly, so if it diverges, the other series must also diverge.

As an example, look at the series

$$\sum_{n=1}^{\infty} \frac{1}{n+1}$$

And compare it with the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Look at the limit of the fraction of corresponding terms:

$$\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

The limit is positive, so the two series converge or diverge together. Since the harmonic series diverges, so does the other series.

As another example,

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

Compared with the harmonic series gives

$$\lim_{n\to\infty} \frac{\frac{1}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1}{n+\frac{1}{n}} = 0$$

Which says that if the harmonic series converges, the first series must also converge. Unfortunately, the harmonic series does not converge, so we must test the series again. Let's try *n*^-2:

$$\lim_{n\to\infty} \frac{\frac{1}{n^2+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1$$

This limit is positive, and *n*^-2 is a convergent *p*-series, so the series in question does converge.

8.3.1.5 The Alternating Series Test

If *a*[*n*]=(-1)^(*n*+1)*b*[*n*], where *b*[*n*] is positive, decreasing, and converging to zero, then the sum of *a*[*n*] converges.

$$a_n = (-1)^{n+1} \cdot b_n, b_n > 0 \text{ for all } n, b_n \text{ decreasing}$$

$$\lim_{n\to\infty} b_n = 0 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

With the Alternating Series Test, all we need to know to determine convergence of the series is whether the limit of *b*[*n*] is zero as n goes to infinity.

So, given the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$$

Look at the limit of the non-alternating part:

$$\lim_{n\to\infty}\frac{1}{n}=0$$

So, this series converges. Note that the other test dealing with negative numbers, the Absolute Convergence Test, would not tell us that this series converges.

8.3.1.6 The Absolute Convergence Test

If the sum of |*a*[*n*]| converges, then the sum of *a*[*n*] converges.

$$\sum_{n=1}^{\infty}|a_n| \text{ converges} \Rightarrow \sum_{n=1}^{\infty}a_n \text{ converges}$$

We call this type of convergence *absolute convergence*.

As an example, look at

$$\sum_{n=1}^{\infty}\frac{\sin n}{n^2}.$$

We know that since the absolute value of sin(*x*) is always less than or equal to one, then

$$\left|\frac{\sin n}{n^2}\right| \le \frac{1}{n^2}$$

So, by the Comparison Test, and the fact that

$$\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Is a convergent *p*-series, we find that

$$\sum_{n=1}^{\infty}\left|\frac{\sin n}{n^2}\right|$$

Converges, so

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

Converges.

8.3.1.7 The Ratio Test

If the limit of |*a* [*n*+1]/*a*[*n*]| is less than 1, then the series (absolutely) converges. If the limit is larger than one, or infinite, then the series diverges.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$

Let's look at an example of this:

$$\sum_{n=1}^{\infty} \frac{3n}{2^n}$$

Look at the ratio of consecutive terms, and find the limit.

$$\lim_{n\to\infty}\left|\frac{\frac{3(n+1)}{2^{n+1}}}{\frac{3n}{2^n}}\right| = \lim_{n\to\infty}\frac{3\,(n+1)\,2^n}{3n\cdot 2^{n+1}} = \lim_{n\to\infty}\frac{n+1}{2n} \stackrel{*}{=} \lim_{n\to\infty}\frac{1}{2} = \frac{1}{2}$$

Note the use of l'Hôpital's Rule in the second-to-last step. This limit, being less than 1, tells us that the series converges.

3.1.8 The Root Test

If the limit of |*a*[*n*]|^ (1/*n*) is less than one, then the series (absolutely) converges. If the limit is larger than one, or infinite, then the series diverges.

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} < 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$

Here's an example of the root test. Look at the series

$$\sum_{n=1}^{\infty} \frac{3n}{2^n}$$

Find the limit of the *n*th root of the *n*th term.

$$\lim_{n\to\infty} \sqrt[n]{\frac{3n}{2^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{3n}}{2} = \frac{1}{2} \lim_{n\to\infty} \exp\left(\frac{1}{n} \ln 3n\right) =$$

$$\frac{1}{2} \exp\left(\lim_{n\to\infty} \frac{\ln 3n}{n}\right) \stackrel{*}{=} \frac{1}{2} \exp\left(\lim_{n\to\infty} \frac{\frac{3}{3n}}{1}\right) = \frac{1}{2} e^0 = \frac{1}{2}$$

Note the use of l'Hôpital's Rule in determining the limit. Since this limit is less than 1, the series converges.

Check in Progress-II

Note: Please give solution of questions in space give below:

Q. 1 Define Ratio Test.

Solution:

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Q. 2 Define Root Test.

Solution:

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8.4 START OF TESTS

Notes

Look at your series. Treat the individual terms of the series as a sequence instead. Does this sequence converge to zero? If not, then the series does not converge. This is called the Divergence Test. If the limit is zero, then continue with the tests. If it's difficult to tell, then continue with the tests.

For example,

$$\sum_{n=1}^{\infty} n^2 \text{ diverges since } \lim_{n\to\infty} n^2 \neq 0$$

Try to Find the Form of the Series Terms

Try to determine the form of the terms of the series. Pick the type that best seems to fit your series.

- Type A: *n* never appears in a power.
- Type B: *n* only appears in a power.
- Type C: *n* appears in both a power and a base.
- Type D: *n*! Appears.

Some examples:

Type A $\dfrac{n^3 + \sqrt{n}}{2n^2 \sqrt[8]{3n + 2n^2}}$

Type B $\dfrac{2^{n+2}}{3^{n+1}}$

Type C $\left(\dfrac{3n^3 + 2n - 1}{2n^3 + \frac{1}{n}}\right)^n$

Type D $\dfrac{n!}{n^n}$

No *n* Powers

If some of the terms are negative, then look at the series of absolute values instead. If that converges, then the original will as well, because of the Absolute Convergence Test. Try to get rid of negative exponents, and treat them as fractions instead. If you have fractions inside of fractions, try finding common denominators and reduce to just one fraction.

As an example,

$$\frac{1}{n+1}\frac{3n^{-2}}{(2n)^{-1}\sqrt{n}} = \frac{6n}{n^2\,(n+1)\sqrt{n}} = \frac{6}{(n^2+n)\sqrt{n}} \leq 6\cdot\frac{1}{n^2}$$

Then use the Comparison Test, comparing with the *p*-series above:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\text{ converges, so }\sum_{n=1}^{\infty}\frac{1}{n+1}\frac{3n^{-2}}{(2n)^{-1}\sqrt{n}}\text{ converges}$$

If there aren't any transcendental functions (like the natural logarithm, the tangent function, and so forth) in the term, do the following: find the largest power of *n* contained in the[*n*] term. If the term is a fraction, find the largest power of *n* in both the numerator and the denominator, and subtract the largest power in the denominator from the largest power in the numerator to get the largest power of *n* in the entire term. If two *n* terms are multiplied, add the powers. If a group of terms is inside, say, a cube root, divide all powers inside the cube root by three.

The resulting highest power should be negative. If not, then find the limit of the terms: it probably won't be zero. If it is negative, then try the Comparison Test or the Limit Comparison Test with *b*[*n*] = 1/*n*a*p*, where *p* is the power from above. If that doesn't work, and the term looks like something you can integrate, try the Integral Test.

For instance,

$$\frac{n^3+\sqrt{n}}{2n^2\sqrt[3]{3n+2n^2}}$$

Has a power of 3 in the numerator, and a power of 2+2/3 in the denominator, so the whole fraction should compare favorably with *n*^(1/3). Use the Limit Comparison Test, and divide the above term by *n*^(1/3) to get

$$\frac{n^3+\sqrt{n}}{2n^2\sqrt[3]{3n^2+2n^3}}$$

Then, in order to find the limit as *n* goes to infinity, divide both top and bottom by *n*^3 to get

Notes

$$\frac{1 + n^{-2.5}}{2\sqrt[3]{3n^{-1} + 2}}$$

And then the limit is found to be 2^ (-4/3), a positive number. Since *n*^ (1/3) diverges as a series (see *p*-series), the original series also diverges.

If you have an "expr" function, or a hyperbolic trig function, then write out the functions in terms of *e*, and you'll see that you *do* have an *n* in a power. Try those tests instead.

If you have a logarithm, then try treating it as an extremely small power of *n*. In fact, for any *n*u*ke*, with *k* positive, ln *n* < *n*^*k* for large enough *n*. So,

$$\frac{\ln n}{n^2} < \frac{n^{\frac{1}{2}}}{n^2} = n^{-\frac{3}{2}}$$

Which is a converging *p*-series, so the original series converges as well.

If you have a trigonometric function, check to see if you can find a pattern to the results; this is most likely if you have pi inside the trig function. If you just have a sine or a cosine function, try treating those as if they were constants; that might work, especially with a comparison test (for example, |sin *x*|<=1).

For example, the sum of sin (pi/2**n*) is really 1 + 0 + -1 + 0 + 1 + 0 + -1 + 0 + ... Its sequence of partial sums has no limit, so the series does not converge. The sum of |sin *n*|/*n*^2 has terms smaller than 1/*n*^2, which converges, so the sum of |sin *n*|/*n*^2 also converges.

NS All Over the Place

Change the negative terms to positive by taking the absolute value of the terms. If the new series converges, then so does the old one, by the Absolute Convergence Test.

With an *n* in the power of the term, the Root Test can cancel that power of *n*, so try that. Also, the Ratio Test often can be useful in such circumstances.

For instance, whether or not the sum of

$$\left(\frac{3n^3+2n-1}{2n^3+\frac{1}{n}}\right)^n$$

Converges can be found using the Root Test:

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{3n^3+2n-1}{2n^3+\frac{1}{n}}\right)^n}=\lim_{n\to\infty}\frac{3n^3+2n-1}{2n^3+\frac{1}{n}}=\frac{3}{2}$$

So the series diverges.

Change the negative terms to positive by taking the absolute value of the terms. If the new series converges, then so does the old one, by the Absolute Convergence Test.

With an *n* in the power of the term, the Root Test can cancel that power of *n*, so try that. Also, the Ratio Test often can be useful in such circumstances.

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Converges can be found using the Root Test:

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{3n^3+2n-1}{2n^3+\frac{1}{n}}\right)^n}=\lim_{n\to\infty}\frac{3n^3+2n-1}{2n^3+\frac{1}{n}}=\frac{3}{2}$$

So the series diverges.

None of this has helped.

So, nothing has worked so far yet? Here are a few things to try. If you haven't done so yet, try checking to see if the limit of the terms is zero (the Divergence Test). Try splitting up a term into the sum of two different terms, and checking each separately, but be careful: if they both diverge, then the diverging parts may cancel each other out. If there is

Notes

subtraction, and the parts being subtracted look similar, you might check for a Telescoping Series.

Still nothing? Does the series alternate between positive and negative terms? You might try the Alternating Series Test. And, if you haven't tried it yet, try the Integral Test.

As an example, look at

$$\sum_{n=1}^{\infty} \frac{n^2+2}{n^3+6n}$$

The "highest power" method from the "no *n* powers" page will work, but let's use the Integral Test.

$$\int_1^{\infty} \frac{x^2+2}{x^3+6x}\,dx = \frac{1}{3}\int_1^{\infty} \frac{3x^2+6}{x^3+6x}\,dx = \frac{1}{3}\ln\left(x^3+6x\right)\Big|_1^{\infty} = \infty$$

So, since the corresponding integral doesn't converge, the series won't converge either.

n Only in the Power

f some terms are negative, look at the series of absolute values instead. The Absolute Convergence Test says that if the latter series converges, then so does the former.

Try to isolate the part of the term raised to the *n* power from the part of the series not raised to the *n* power. Consider splitting the series into two pieces based on this, if necessary.

So, the sum of 3*(1/2^*n*) is 3 times the sum of 1/2^*n*. The sum of 1/2^*n* converges, so 3 times is also converges. Similarly, the sum of 3+1/2^*n* equals the sum of 3 + the sum of 1/2^*n*. Since the sum of 3 diverges, and the sum of 1/2^*n* converges, the series diverges. You have to be careful here, though: if you get a sum of two diverging series, occasionally they will cancel each other out and the result will converge.

If the power is *n*+1 or such, then factor out terms until you just have an *n* power. Then try to match up the term with the Geometric Series: use the part raised to the *n* power as *r*, and the part not raised to the *n* power as *a*. If it doesn't fit exactly, see if you can use a Comparison Test or Limit Comparison Test.

For instance,

$$\frac{2^{n+2}}{3^{n+1}} = \frac{4 \cdot 2^n}{3 \cdot 3^n} = \frac{4}{3} \cdot \left(\frac{2}{3}\right)^n$$

The sum of which converges to 4/3*(1/ (1-2/3)) = 4.

It also may be worthwhile to try the Root Test, since taking an *n*th root will conveniently rid the term of an *n*th power. Also, you will often get a lot of cancellation using the Ratio Test.

As an example, look at the sum of (1/3) ^ (*n*+2). Using the Ratio Test, we get

$$\frac{\left(\frac{1}{3}\right)^{n+3}}{\left(\frac{1}{3}\right)^{n+2}} = \frac{1}{3} < 1$$

So the series converges.

ns All Over the Place

Change the negative terms to positive by taking the absolute value of the terms. If the new series converges, then so does the old one, by the Absolute Convergence Test.

With an *n* in the power of the term, the Root Test can cancel that power of *n*, so try that. Also, the Ratio Test often can be useful in such circumstances.

For instance, whether or not the sum of

$$\left(\frac{3n^3 + 2n - 1}{2n^3 + \frac{1}{n}}\right)^n$$

Converges can be found using the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{3n^3+2n-1}{2n^3+\frac{1}{n}}\right)^n} = \lim_{n\to\infty} \frac{3n^3+2n-1}{2n^3+\frac{1}{n}} = \frac{3}{2}$$

So the series diverges.

Factorials

f some terms are negative, take absolute values. The Absolute Convergence Test allows this: if the new series converges, then the old one will also.

The Ratio Test will usually cause a lot of cancellation in these cases; cancellation which will rid you of most, if not all, of the factorial part.

For instance, look at:

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

The Ratio Test gives

(n+1)!/(n+1)^{n+1}

----

n!/n^n

(n+1)! n^n

=----------

n!(n+1)^{n+1}

(n+1) n^n

=-----------

(n+1) (n+1)^n

n^n/(n+1)^n = [n/(n+1)]^n --> 1/e as n->infinity

The reason is that [(n+1)/n)]^n=[1+1/n]^n -> e as n->infinity. This is by definition of the exponential.

Hence, according to the Ratio Test the original series is convergent.

Theorem (Comparison Test). Let $\sum_{n=1}^{\infty} M_n$ be a convergent series of real nonnegative terms. If $\{z_n\}$ is a sequence of complex numbers and $|z_n| \le M_n$ holds for all n, then $\sum_{n=1}^{\infty} z_n$ converges.

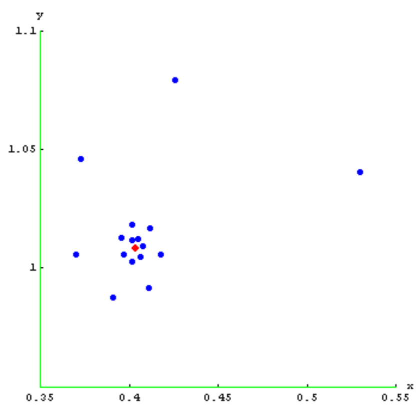
Corollary If $\sum_{n=1}^{\infty} |z_n|$ converges, then $\sum_{n=1}^{\infty} z_n$ converges.

In other words, absolute convergence implies convergence for complex series as well as for real series.

Example Show that the series $\sum_{n=1}^{\infty} \frac{(3+4i)^n}{5^n n^2}$ is convergent.

Solution. We calculate $|z_n| = \left|\frac{(3+4i)^n}{5^n n^2}\right| = \frac{1}{n^2} = M_n$. Using the comparison test and the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we determine that $\sum_{n=1}^{\infty} \left|\frac{(3+4i)^n}{5^n n^2}\right|$ converges and hence, by Corollary 4.1, so does $\sum_{n=1}^{\infty} \frac{(3+4i)^n}{5^n n^2}$.

Aside. Just for fun, we can graph some of the partial sums of this complex series.



The partial sums $\{S_n\} = \left\{\sum_{k=1}^{n} \frac{(3+4i)^k}{5^k k^2}\right\}$ converge to the value

$S \approx 0.403311 + 1.00841\, i$

Notes

Let *b*[*n*] be a second series. Require that all *a*[*n*] and *b*[*n*] are positive. If *b*[*n*] converges, and *a*[*n*]<=*b*[*n*] for all *n*, then *a*[*n*] also converges. If the sum of *b*[*n*] diverges, and *a*[*n*]>=*b*[*n*] for all *n*, then the sum of *a*[*n*] also diverges.

$$a_n, b_n > 0 \text{ for all } n$$

$$\sum_{n=1}^{\infty} b_n \text{ converges}, a_n \leq b_n \text{ for all } n \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

$$\sum_{n=1}^{\infty} b_n \text{ diverges}, a_n \geq b_n \text{ for all } n \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$

The idea with this test is that if each term of one series is smaller than another, then the sum of that series must be smaller. So, if every term of a series is smaller than the corresponding term of a converging series, the smaller series must also converge. And if a smaller series diverges, the larger one must also diverge.

As an example, consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n+1}.$$

Compare that with a second series as follows:

$$\frac{1}{n+1} > \frac{1}{2n} \text{ (since } n\text{+1<2}n \text{ for } n\text{>=1)} = \frac{1}{2} \cdot \frac{1}{n}.$$

$$\sum_{n=1}^{\infty} \frac{1}{2} \cdot \frac{1}{n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}.$$

Since this new, smaller sum diverges (it is a harmonic series), the original sum also diverges.

For another example, look at

$$\sum_{n=1}^{\infty} \frac{n-1}{n^3}.$$

Compare that with a second series also:

$$\frac{n-1}{n^3} = \frac{n}{n^3} - \frac{1}{n^3} < \frac{1}{n^2}.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges (since it is a *p*-series with *p* greater than one), so the first sum also converges.

Check in Progress-III

Note : Please give solution of questions in space give below:

Q. 1 Define Comparison Test.

Solution:

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…………………………………………………………………………………

…..………………………………………………………………………………

………..……………………………………………………………………

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Q. 2 Show that the series $\sum_{n=1}^{\infty} \frac{(3+4i)^n}{5^n n^2}$ is convergent.

Solution:

…………………………………………………………………………………

…………………………………………………………………………………

…..………………………………………………………………………………

………..

8.4.1 Examples

Exercise 1. Find the following limits.

1 (a). $\lim_{n\to\infty} \left(\frac{1+i}{2}\right)^n$.

Solution 1 (a).

Notes

Answer $\lim_{n\to\infty}\left(\frac{1+i}{2}\right)^n = 0$.

Solution. Method I. We have

$$z_n = \left(\frac{1+i}{2}\right)^n$$

$$= x_n + i\,y_n$$

$$= \left(\frac{\sqrt{2}}{2}\right)^n \cos\frac{n\pi}{4} + i\left(\frac{\sqrt{2}}{2}\right)^n \sin\frac{n\pi}{4}$$

We can use the "squeeze theorem" for real sequences to show that both $x_n = \left(\frac{\sqrt{2}}{2}\right)^n \cos\frac{n\pi}{4}$ and $y_n = \left(\frac{\sqrt{2}}{2}\right)^n \sin\frac{n\pi}{4}$ converge to 0.

$$0 \le \lim_{n\to\infty}\left|\left(\frac{\sqrt{2}}{2}\right)^n \cos\frac{n\pi}{4}\right| \le \lim_{n\to\infty}\left(\frac{\sqrt{2}}{2}\right)^n = 0$$, and

$$0 \le \lim_{n\to\infty}\left|\left(\frac{\sqrt{2}}{2}\right)^n \sin\frac{n\pi}{4}\right| \le \lim_{n\to\infty}\left(\frac{\sqrt{2}}{2}\right)^n = 0$$,

Hence we have, $\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(\frac{\sqrt{2}}{2}\right)^n \cos\frac{n\pi}{4} = 0$ and $\lim_{n\to\infty} y_n = \lim_{n\to\infty}\left(\frac{\sqrt{2}}{2}\right)^n \sin\frac{n\pi}{4} = 0$.

Therefore $\lim_{n\to\infty}\left(\frac{1}{2}+\frac{i}{2}\right)^n = \lim_{n\to\infty} x_n + i\lim_{n\to\infty} y_n = 0$.

Solution. Method II. Use the fact that $\lim_{n\to\infty} z_n = 0$ iff $\lim_{n\to\infty} |z_n| = 0$. (You will be asked to prove this fact in Exercise 17.)

Find the limit of the sequence of absolute values:

$$\lim_{n\to\infty} \left| z_n \right| = \lim_{n\to\infty} \left| \frac{1}{2} + \frac{i}{2} \right|^n$$

$$= \lim_{n\to\infty} \left(\frac{\sqrt{2}}{2} \right)^n$$

$$= 0$$

Then, $\lim_{n\to\infty} \left| \frac{1}{2} + \frac{i}{2} \right|^n = 0$ implies that $\lim_{n\to\infty} \left(\frac{1}{2} + \frac{i}{2} \right)^n = 0$.

We are done.

Exercise 1. Find the following limits.

1 (b). $\lim_{n\to\infty} \frac{n + (i)^n}{n}$.

Solution 1 (b).

See text and/or instructor's solution manual.

Answer $\lim_{n\to\infty} \frac{n + (i)^n}{n} = 1$.

Solution. Rewrite the series as follows:

$$\lim_{n\to\infty} \frac{n + (i)^n}{n} = \lim_{n\to\infty} \left(\frac{n}{n} + \frac{(i)^n}{n} \right)$$

$$= \lim_{n\to\infty} \left(1 + \frac{(i)^n}{n} \right)$$

$$= 1 + \lim_{n\to\infty} \frac{(i)^n}{n}$$

This boils down to showing that $\lim_{n\to\infty} \frac{(i)^n}{n} = 0$.

Notes

Method I. We have

$$z_n = \frac{(i)^n}{n}$$

$$= x_n + i\, y_n$$

$$= \frac{1}{n} \cos \frac{n\pi}{2} + i \frac{1}{n} \sin \frac{n\pi}{2}$$

We can use the "squeeze theorem" for real sequences to show that both $x_n = \frac{1}{n} \cos \frac{n\pi}{2}$ and $y_n = \frac{1}{n} \sin \frac{n\pi}{2}$ converge to 0.

$$0 \le \lim_{n\to\infty} \left| \frac{1}{n} \cos \frac{n\pi}{2} \right| \le \lim_{n\to\infty} \frac{1}{n} = 0,$$ and

$$0 \le \lim_{n\to\infty} \left| \frac{1}{n} \sin \frac{n\pi}{2} \right| \le \lim_{n\to\infty} \frac{1}{n} = 0,$$

Hence we have,

$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n} \cos \frac{n\pi}{2} = 0$ and $\lim_{n\to\infty} y_n = \frac{1}{n} \sin \frac{n\pi}{2} = 0$.

Therefore $\lim_{n\to\infty} \frac{(i)^n}{n} = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = 0$.

Method II. Use the fact that $\lim_{n\to\infty} z_n = 0$ iff $\lim_{n\to\infty} | z_n | = 0$.
(You will be asked to prove this fact in Exercise 17.)

Find the limit of the sequence of absolute values:

$$\lim_{n\to\infty} \left| z_n \right| = \lim_{n\to\infty} \left| \frac{(i)^n}{n} \right| = \lim_{n\to\infty} \frac{1}{n} = 0,$$

From Method I or Method II, we have established the fact that

$$\lim_{n\to\infty} \left| \frac{(i)^n}{n} \right| = 0.$$

Then, $\lim_{n\to\infty}\left|\frac{(i)^n}{n}\right| = 0$ implies that $\lim_{n\to\infty}\frac{(i)^n}{n} = 0$.

$$\lim_{n\to\infty}\frac{n+(i)^n}{n} = \lim_{n\to\infty}\left(1+\frac{(i)^n}{n}\right)$$

$$= 1 + \lim_{n\to\infty}\frac{(i)^n}{n}$$

$$= 1 + 0$$

Therefore,

$$= 1$$

We are done.

Exercise 1. Find the following limits.

1 (c). $\lim_{n\to\infty}\frac{n^2 + i\,2^n}{2^n}$.

Solution 1 (c).

.Answer $\lim_{n\to\infty}\frac{n^2 + i\,2^n}{2^n} = i$.

Solution. The limit of the sequence can be computed as follows:

$$\lim_{n\to\infty}\frac{n^2 + i\,2^n}{2^n} = \lim_{n\to\infty}\left(\frac{n^2}{2^n} + i\,\frac{2^n}{2^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n^2}{2^n} + i\right)$$

$$= \lim_{n\to\infty}\frac{n^2}{2^n} + i\lim_{n\to\infty} 1$$

$$= 0 + i$$

$$= i$$

Notes

The sequence $\left\{\frac{n^2}{2^n}\right\}$ was studied in calculus and $\lim_{n\to\infty} \frac{n^2}{2^n} = 0$, can be established by using either the comparison test or L'Hôpital's rule.

We are done.

Exercise 1. Find the following limits.

1 (d). $\lim_{n\to\infty} \frac{(n + i)(1 + n i)}{n^2}$.

Solution 1 (d).

Answer $\lim_{n\to\infty} \frac{(n+i)(1+n i)}{n^2} = i$.

Solution. Expand the formula for the terms in the sequence

$$\frac{(n+i)(1+n i)}{n^2} = \frac{(n+n i^2)+i(1+n^2)}{n^2} = i \frac{n^2+1}{n^2}.$$

Therefore, $\lim_{n\to\infty} \frac{(n+i)(1+n i)}{n^2} = i \lim_{n\to\infty} \frac{n^2+1}{n^2} = i$.

We are done.

Exercise 2. Show that $\lim_{n\to\infty} (i)^{\frac{1}{n}} = 1$, where $(i)^{\frac{1}{n}}$ is the principal value of the n^{th} root of i.

Solution. The principal n^{th} root was introduced

$$\lim_{n\to\infty} (i)^{\frac{1}{n}} = \lim_{n\to\infty} \left(e^{i\frac{\pi}{2}}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} e^{i\frac{\pi}{2n}}$$

$$= \lim_{n\to\infty} \left(\cos\frac{\pi}{2n} + i\sin\frac{\pi}{2n}\right)$$

$$= \lim_{n\to\infty} \cos\frac{\pi}{2n} + i\lim_{n\to\infty} \sin\frac{\pi}{2n}$$

$$= \cos 0 + i\sin 0$$

$$= 1 + i\,0$$

$$= 1$$

We are done.

Exercise 3. Suppose that $\lim_{n\to\infty} z_n = z_0$. Show that $\lim_{n\to\infty} \overline{z_n} = \overline{z_0}$.

Solution. Method I.

Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} z_n = z_0$, there exists N_ϵ such that if $n > N_\epsilon$ then $z_n \in D_\epsilon(z_0)$, i.e., $|z_n - z_0| < \epsilon$.

But since $|\overline{z_n} - \overline{z_0}| = |z_n - z_0|$, this implies that if $n > N_\epsilon$, then $\overline{z_n} \in D_\epsilon(\overline{z_0})$. Therefore $\lim_{n\to\infty} \overline{z_n} = \overline{z_0}$.

Solution. Method II.

Let $z_n = x_n + i\,y_n$, with $\lim_{n\to\infty} z_n = z_0$, and $\lim_{n\to\infty} (x_n + i\,y_n) = x_0 + i\,y_0$

.

Both $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ exist, and

$\lim_{n\to\infty} x_n = x_0$, and

$\lim_{n\to\infty} y_n = y_0$.

Notes

Thus, $\lim_{n\to\infty}(-y_n)$ exists, and $\lim_{n\to\infty}(-y_n) = -y_0$.

Therefore,

$$\lim_{n\to\infty}\overline{z_n} = \lim_{n\to\infty}(x_n - i\,y_n)$$

$$= \lim_{n\to\infty}(x_n + i\,(-y_n))$$

$$= \lim_{n\to\infty}x_n + i\lim_{n\to\infty}(-y_n)$$

$$= x_0 + i\,(-y_0)$$

$$= x_0 - i\,y_0$$

$$= \overline{z_0}$$

Exercise 4. Suppose that the complex sequence $\{z_n\}$ converges to ξ. Show that $\{z_n\}$ is bounded in two ways.

4 (a). Write $z_n = x_n + i\,y_n$ and use the fact that convergent series of real numbers are bounded.

Solution. Let $z_n = x_n + i\,y_n$, with $\lim_{n\to\infty} z_n = z_0$,

and $\lim_{n\to\infty}(x_n + i\,y_n) = x_0 + i\,y_0$.

$\lim_{n\to\infty} x_n = x_0$ And $\lim_{n\to\infty} y_n = y_0$.

Since $\{x_n\}$ is a convergent sequence of real numbers, there exists a real number $R_1 > 0$ such that $|x_n| < R_1$ for all n.

Since $\{y_n\}$ is a convergent sequence of real numbers, there exists a real number $R_2 > 0$ such that $|y_n| < R_2$ for all n.

Let $R = R_1 + R_2$. Then

$$|z_n| = |x_n + i\,y_n| \le |x_n| + |y_n| < R_1 + R_2 = R.$$

Therefore, the sequence $\{z_n\}$ is bounded.

8.5 SUMMARY

In this unit we study convergence test and its examples with solution. We study comparison test of series. We study p-series and Geometric series. We study divergent test, convergence test, limit comparison test, alternating series test, absolute convergence test, and ratio test with examples. We study some exercise for series test.

8.6 KEYWORD

BOUNDED:(of an object) rebound from a surface

ABSOLUTE:viewed or existing independently and not in relation to other things; not relative or comparative

DIVERGENCE : the scalar product of the operator del and a given vector, which gives a measure of the quantity of flux emanating from any point of the vector field or the rate of loss of mass, heat, etc., from it

8.7 QUESTIONS FOR REVIEW

Q. 1 Show that $\sum_{k=0}^{\infty}\left(\frac{1}{k+1+i}-\frac{1}{k+i}\right) = i$.

Q. 2 Suppose that $\sum_{n=1}^{\infty} z_n = S$. Show that $\sum_{n=1}^{\infty} \overline{z_n} = \overline{S}$.

Q. 3 Does $\lim_{n\to\infty}\left(\frac{1+i}{\sqrt{2}}\right)^n$ exist? Why?

Q. 4 Let $z_n = r_n e^{i\theta_n}$, where $\theta_n = \mathrm{Arg}[z_n]$.

Q. 5 Is it possible to have $\lim_{n\to\infty} z_n = z_0 = r_0 e^{i\theta_0}$, but $\lim_{n\to\infty} r_n$ does not exist?

Q. 6 Show that, if $\sum_{n=1}^{\infty} z_n$ converges, then $\lim_{n\to\infty} z_n = 0$.

Q. 7 State whether the following series converge or diverge. Justify your answers.

(a). $\sum_{n=1}^{\infty} \frac{(i)^n}{n}$. **(b).** $\sum_{n=1}^{\infty} \left(\frac{1}{n} + \frac{i}{2^n}\right)$

Q. 8 If $\sum_{k=0}^{\infty} z_k$ converges, show that $\left|\sum_{k=0}^{\infty} z_k\right| \le \sum_{k=0}^{\infty} \left|z_k\right|$.

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8.9 ANSWER TO CHECK YOUR PROGRESS

Check In Progress-I

Answer Q. 1 Check in Section 1.2

2 Check in Section 1.3

Check In Progress-II

Answer Q. 1 Check in section 3.1.7

2 Check in Section 3.1.8

Check In Progress-III

Answer Q. 1 Check in section 4

2 Check in Section 4

UNIT 9 : ABSOLUTE AND UNIFORM CONVERGENCE

STRUCTURE

9.0 Objective

9.1 Introduction

9.1.1 Absolute Convergent

9.1.2 Relation to Convergence

9.1.3 Convergent Series

9.1.4 Uniform Convergence

9.2 Abel's Convergence Theorem

9.2.1 Abel's Uniform Convergence Test

9.2.2 Weiertrass M-test

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9.0 OBJECTIVES

- Deals with absolute convergence
- Deals with uniform convergence
- Study with convergent series
- Deals with Weierstrass M-test
- Deals with Abel's uniform convergence test

9.1 INTRODUCTION

When we first talked about series convergence we briefly mentioned a stronger type of convergence but didn't do anything with it because we didn't have any tools at our disposal that we could use to work problems involving it. We now have some of those tools so it's now time to talk about absolute convergence in detail.

Describes a series that converges when all terms are replaced by their absolute values. To see if a series converges absolutely, replace any subtraction in the series with addition. If the new series converges, then the original series converges absolutely.

Note: Any series that converges absolutely is itself convergent.

9.1.1 Absolute Convergent

A series $\sum_n u_n$ is said to converge absolutely if the series $\sum_n |u_n|$ converges, where $|u_n|$ denotes the absolute value. If a series is absolutely convergent, then the sum is independent of the order in which terms are summed. Furthermore, if the series is multiplied by another absolutely convergent series, the product series will also converge absolutely.

Definition

A series ∑an is called absolutely convergent if ∑|an| is convergent.
If ∑an is convergent and ∑|an| is divergent we call the series conditionally convergent.

We also have the following fact about absolute convergence.

Fact

If ∑an is absolutely convergent then it is also convergent.

Proof

First notice that |an| is either an or it is −an depending on its sign. This means that we can then say, 0≤an+|an|≤2|an|

Now, since we are assuming that ∑|an| is convergent then ∑2|an| is also convergent since we can just factor the 2 out of the series and 2 times a finite value will still be finite. This however allows us to use the Comparison Test to say that $\sum(a_n+|a_n|)$ is also a convergent series.

Finally, we can write, $\sum a_n=\sum(a_n+|a_n|)-\sum|a_n|$

and so $\sum a_n$ is the difference of two convergent series and so is also convergent.

This fact is one of the ways in which absolute convergence is a "stronger" type of convergence. Series that are absolutely convergent are guaranteed to be convergent. However, series that are convergent may or may not be absolutely convergent.

Let's take a quick look at a couple of examples of absolute convergence.

Roughly speaking there are two ways for a series to converge: As in the case of $\sum 1/n^2$, the individual terms get small very quickly, so that the sum of all of them stays finite, or, as in the case of $\sum(-1)^{n-1}/n$, the terms don't get small fast enough ($\sum 1/n$ diverges), but a mixture of positive and negative terms provides enough cancellation to keep the sum finite. You might guess from what we've seen that if the terms get small fast enough to do the job, then whether or not some terms are negative and some positive the series converges.

9.1.2 Relation to Convergence

If *G* is complete with respect to the metric *d*, then every absolutely convergent series is convergent. The proof is the same as for complex-valued series: use the completeness to derive the Cauchy criterion for convergence—a series is convergent if and only if its tails can be made arbitrarily small in norm—and apply the triangle inequality.

In particular, for series with values in any Banach space, absolute convergence implies convergence. The converse is also true: if absolute convergence implies convergence in a normed space, then the space is a Banach space.

If a series is convergent but not absolutely convergent, it is called conditionally convergent. An example of a conditionally convergent series is the alternating harmonic series. Many standard tests for divergence and convergence, most notably including the ratio test and the root test, demonstrate absolute convergence. This is because a power series is absolutely convergent on the interior of its disk of convergence.

9.1.3 Convergent Series

A series is said to be convergent if it approaches some limit (D'Angelo and West 2000, p. 259).

Formally, the infinite series $\sum_{n=1}^{\infty} a_n$ is convergent if the sequence of partial sums

$$S_n = \sum_{k=1}^{n} a_k \tag{1}$$

Is convergent. Conversely, a series is divergent if the sequence of partial sums is divergent. If $\sum u_k$ and $\sum v_k$ are convergent series, then $\sum (u_k + v_k)$ and $\sum (u_k - v_k)$ are convergent. If $c \neq 0$, then $\sum u_k$ and $c \sum u_k$ both converge or both diverge. Convergence and divergence are unaffected by deleting a finite number of terms from the beginning of a series. Constant terms in the denominator of a sequence can usually be deleted without affecting convergence. All but the highest power terms in polynomials can usually be deleted in both numerator and denominator of a series without affecting convergence.

If the series formed by taking the absolute values of its terms converges (in which case it is said to be absolutely convergent), then the original series converges.

Conditions for convergence of a series can be determined in the Wolfram Language using Sum Convergence [*a*, *n*].

The series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} = \infty \tag{2}$$

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n (\ln \ln n)} = \infty \tag{3}$$

both diverge by the integral test, although the latter requires a googolplex number of terms before the partial sums exceed 10 (Zwillinger 1996, p. 39). In contrast, the sums

$$\sum_{n=2}^{\infty} \frac{1}{n (\ln n)^2} \approx 2.109742801236 \tag{4}$$

(Baxley 1992; Braden 1992; Zwillinger 1996, p. 39; Kreminski 1997; OEIS A115563) and

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n (\ln \ln n)^2} \approx 38.4067680928 \ldots \tag{5}$$

9.1.4 Uniform Convergence

Definition (Uniform Convergence), The sequence $\{S_n(z)\}$ converges uniformly to f(z) on the set T if for every $\epsilon > 0$, there exists a positive integer N_ϵ (depending only on ϵ) such that

if $n \geq N$, then $|S_n(z) - f(z)| < \epsilon$ for all $z \in T$.

If $S_n(z)$ is the n^{th} partial sum of the series $\sum_{k=0}^{\infty} c_k (z-\alpha)^k$, we say that the series $\sum_{k=0}^{\infty} c_k (z-\alpha)^k$ converges uniformly to f (z) on the set T.

A sequence of functions $\{f_n\}$, $n = 1$ 2, 3, is said to be uniformly convergent to f for a set E of values of x if, for each $\epsilon > 0$, an integer N can be found such that

$$|f_n(x) - f(x)| < \epsilon \tag{1}$$

For $n \geq N$ and all $x \in E$.

A series $\sum f_n(x)$ converges uniformly on E if the sequence $\{S_n\}$ of partial sums defined by

$$\sum_{k=1}^{n} f_k(x) = S_n(x) \tag{2}$$

Converges uniformly on E.

To test for uniform convergence, use Abel's uniform convergence test or the Weierstrass M-test. If individual terms $u_n(x)$ of a uniformly converging series are continuous, then the following conditions are satisfied.

1. The series sum

$$f(x) = \sum_{n=1}^{\infty} u_n(x) \tag{3}$$

Is continuous.

2. The series may be integrated term by term

$$\int_a^b f(x)\,dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx. \tag{4}$$

For example, a power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is uniformly convergent on any closed and bounded subset inside its circle of convergence.

3. The situation is more complicated for differentiation since uniform convergence of $\sum_{n=1}^{\infty} u_n(x)$ does not tell anything about convergence of $\sum_{n=1}^{\infty} \frac{d}{dx} u_n(x)$. Suppose that $\sum_{n=1}^{\infty} u_n(x_0)$ converges for some $x_0 \in [a, b]$, that each $u_n(x)$ is differentiable on $[a, b]$, and

that $\sum_{n=1}^{\infty} \frac{d}{dx} u_n(x)$ converges uniformly on $[a, b]$.

Then $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on $[a, b]$ to a function f, and for each $x \in [a, b]$,

$$\frac{d}{dx} f(x) = \sum_{n=1}^{\infty} \frac{d}{dx} u_n(x).$$

Uniform Convergence

Complex functions are the key to unlocking many of the mysteries encountered when power series are first introduced in a calculus course. We begin by discussing an important property associated with power series-uniform convergence.

Recall that, for a function f(z) defined on a set T, the sequence of functions $\{S_n(z)\}$ converges to the function f(z) at the point $z_0 \in T$ provided that $\lim_{n\to\infty} S(z_0) = f(z_0)$. Thus, for the particular point z_0, we know that for each $\epsilon > 0$, there exists a positive integer N_{ϵ,z_0} (which depends on both ϵ and z_0) such that

if $n \geq N_{\epsilon,z_0}$, then $|S(z_0) - f(z_0)| < \epsilon$.

If $S_n(z)$ is the n^{th} partial sum of the series $\sum_{k=0}^{\infty} c_k (z-\alpha)^k$, Statement becomes

if $n \geq N_{\epsilon,z_0}$, then $\left| \sum_{k=0}^{n-1} c_k (z_0-\alpha)^k - f(z_0) \right| < \epsilon$.

For a given value of ϵ, the integer N_{ϵ,z_0} needed to satisfy Statement often depends on our choice of z_0. This is not the case if the sequence $\{S_n(z)\}$ converges uniformly. For a uniformly convergent sequence, it is possible to find an integer N_ϵ (depending only on ϵ) that guarantees Statement no matter what value for $z_0 \in T$ we pick. In other words, if n is large enough, the function $S_n(z)$ is uniformly close to the function f(z) for all $z \in T$. Formally, we have the following definition.

Example . The sequence $\{S_n(z)\} = \left\{e^z + \frac{1}{n}\right\}$ converges uniformly to the function $f(z) = e^z$ on the entire complex plane because for any $\epsilon > 0$, statement is satisfied for all z for $n \geq N_\epsilon$, where N_ϵ is any integer greater than $\frac{1}{\epsilon}$. We leave the details of showing this result as an exercise.

A good example of a sequence of functions that does not converge uniformly is the sequence of partial sums comprising the geometric

9.2 ABEL'S CONVERGENCE THEOREM

Given a Taylor series

$$f(z) = \sum_{n=0}^{\infty} C_n z^n = \sum_{n=0}^{\infty} C_n r^n e^{i n \theta}, \quad (1)$$

Where the complex number z has been written in the polar form $z = r e^{i\theta}$, examine the real and imaginary parts

$$u(r, \theta) = \sum_{n=0}^{\infty} C_n r^n \cos(n\theta) \quad (2)$$

$$v(r, \theta) = \sum_{n=0}^{\infty} C_n r^n \sin(n\theta). \quad (3)$$

Abel's theorem states that, if $u(1, \theta)$ and $v(1, \theta)$ are convergent, then

$$u(1, \theta) + i\, v(1, \theta) = \lim_{r \to 1} f\left(r e^{i\theta}\right). \quad (4)$$

Stated in words, Abel's theorem guarantees that, if a real power series converges for some positive value of the argument, the domain of uniform convergence extends at least up to and including this point. Furthermore, the continuity of the sum function extends at least up to and including this point.

9.2.1 Abel's Uniform Convergence Test

Let $\{u_n(x)\}$ be a sequence of functions. If

1. $u_n(x)$ Can be written $u_n(x) = a_n f_n(x)$,

2. $\sum a_n$ is convergent,

3. $f_n(x)$ Is a monotonic decreasing sequence (i.e., $f_{n+1}(x) \le f_n(x)$) for all n, and

4. $f_n(x)$ Is bounded in some region (i.e., $0 \le f_n(x) \le M$ for all $x \in [a, b]$)

Then, for all $x \in [a, b]$, the series $\sum u_n(x)$ converges uniformly.

9.2.2 Weierstrass M-Test

Let $\sum_{n=1}^{\infty} u_n(x)$ be a series of functions all defined for a set E of values of x. If there is a convergent series of constants

$$\sum_{n=1}^{\infty} M_n,$$

Such that

$$|u_n(x)| \le M_n$$

For all $x \in E$, then the series exhibits absolute convergence for each $x \in E$ as well as uniform convergence in E.

Exercise 1. This exercise relates to Below Figure.

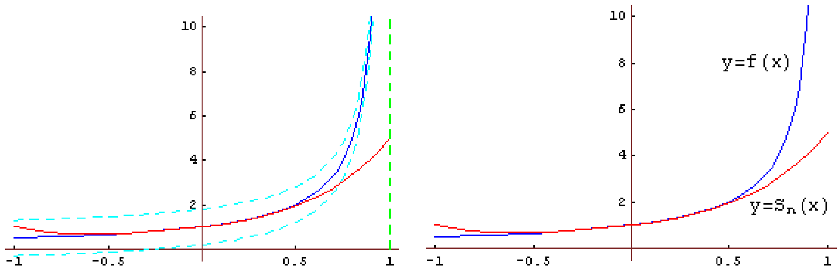


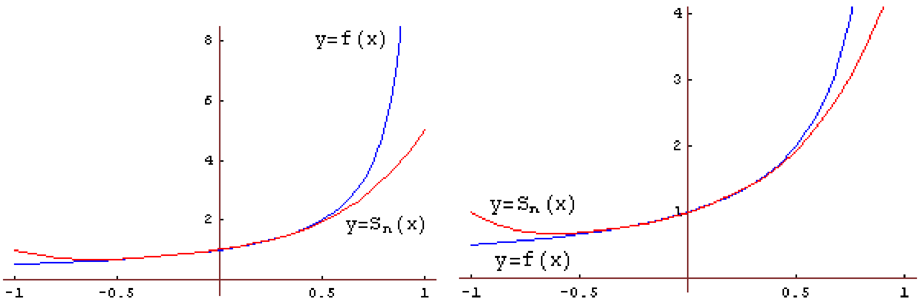
Figure The geometric series does not converge uniformly on $(-1, 1)$ for x near -1, is the graph of $S_n(x) = \sum_{k=0}^{n-1} x^k$ above or below $f(x)$? Explain.

Solution. By definition, $f(x) = \frac{1}{1-x}$ so that $f(-1) = \frac{1}{1-(-1)} = \frac{1}{2}$.

It appears from the graph that the value of the upper function $S_n(x)$ (in red) is approximately $S_n(-1) = 1$,

(certainly larger than $\frac{1}{2}$, so the graph of $S_n(x)$ must be above the graph of $f(x)$ (in blue).

Thus, for x near -1, the graph of $y = S_n(x) = \sum_{k=0}^{n-1} x^k$ is above $y = f(x) = \frac{1}{1-x}$.



The graphs of $y = S_n(x)$ and $y = f(x)$.

We will ask for more details about this situation in Exercise 1 (b).

1 (b). Is the index n in $S_n(x)$ odd or even? Explain.

Solution 1 (b).

See text and/or instructor's solution manual.

Answer. The index n in $S_n(x)$ is odd, i.e. $n = 2k + 1$.

Solution. The sum of the geometric series is

$$S_n(x) = \sum_{k=0}^{n-1} x^k = \frac{1 - x^n}{1 - x} = \frac{1}{1 - x} - \frac{1}{1 - x} x^n .$$

For positive values of x in the interval $0 < x < 1$ we will have

$$S_n(x) = \frac{1}{1 - x} - \frac{1}{1 - x} x^n = f(x) - \frac{1}{1 - x} x^n < f(x) .$$

But for x in the interval $-1 < x < 0$ the sign of x^n will depend on whether n is even or odd.

When $n = 2k$ and $-1 < x < 0$ we have $x^{2k} > 0$ and this implies that

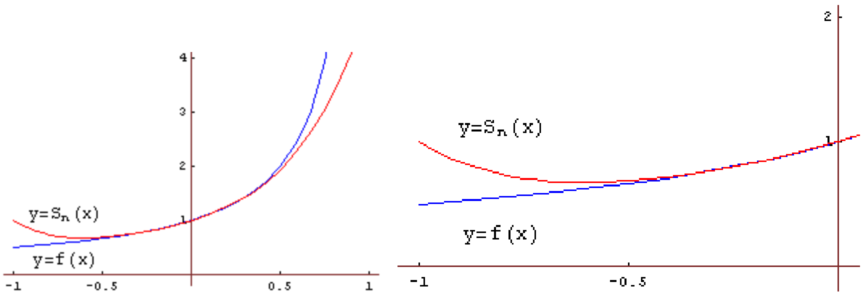
$$S_{2k}(x) = \frac{1}{1-x} - \frac{1}{1-x}x^{2k} = f(x) - \frac{1}{1-x}x^{2k} < f(x).$$

When $n = 2k+1$ and $-1 < x < 0$ we have $x^{2k+1} < 0$ and this implies that

$$S_{2k+1}(x) = \frac{1}{1-x} - \frac{1}{1-x}x^{2k+1} = f(x) - \frac{1}{1-x}x^{2k+1} > f(x).$$

Since the graph of $S_n(x)$ shows $S_n(x) > f(x)$ for $-1 < x < 0$.

Therefore, we conclude that index n in $S_n(x)$ is odd, i.e. $n = 2k+1$.



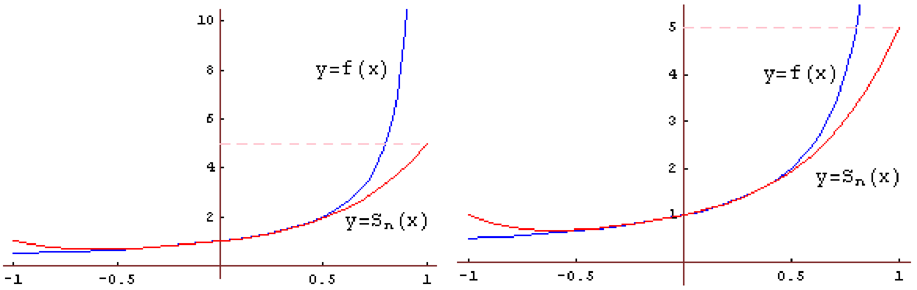
The graphs of $y = S_n(x)$ and $y = f(x)$.

1 (c). Assuming that the graph is accurate to scale, what is the value of n in $S_n(x)$? Explain.

Solution 1 (c).

Solution. From the graph, we approximate $S_5(1) = 5$. Using $S_n(x) = \sum_{k=0}^{n-1} x^k$,

we observe that $1 + 1 + 1^2 + 1^3 + 1^4 = 5$ and deduce that $n = 5$.



The graphs of $y = S_n(x)$ and $y = f(x) = \frac{1}{1-x}$.

It is revealed that $y = S_5(x) = \sum_{k=0}^{5-1} x^k = 1 + x + x^2 + x^3 + x^4$.

Exercise 2. Prove that the following series converge uniformly on the sets indicated.

2 (a). $\sum_{k=1}^{\infty} \frac{1}{k^2} z^k$ Converge uniformly on $\overline{D_1}(0) = \{z : |z| \le 1\}$.

Solution 2 (a).

Solution. Here $u_k(z) = \frac{1}{k^2} z^k$, and for $z \in \overline{D_1}(0) = \{z : |z| \le 1\}$ we have

$$|u_k(z)| = \left| \frac{1}{k^2} z^k \right| \le \frac{1}{k^2} = M_k,$$

the series

$$\sum_{k=1}^{\infty} M_k = \sum_{k=1}^{\infty} \frac{1}{k^2},$$

is known to be convergent (because $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is convergent when $p > 1$).

Therefore, by the Weierstrass M-test, the series $\sum_{k=1}^{\infty} \frac{1}{k^2} z^k$ converges uniformly on $\overline{D_1}(0) = \{z : |z| \le 1\}$

Exercise 2. Prove that the following series converge uniformly on the sets indicated.

2 (b). $\sum_{k=0}^{\infty} \frac{1}{(z^2-1)^k}$ Converge uniformly on $\{z : |z| \geq 2\}$.

Solution 2 (b).

Solution. Here the terms in the series are $u_k(z) = \frac{1}{(z^2-1)^k}$.

If $|z| \geq 2$, then we have $|z|^2 \geq 4$. Recall formula (1-24) in Section 1.3.

(1-24) $|z_1 + z_2| \geq |z_1| - |z_2|$.

In formula (1-24) we set $z_1 = z^2$ and $z_2 = -1$ and get

$$|z^2 - 1| \geq |z^2| - |-1| \geq 4 - 1 = 3.$$

For $z \in \{z : |z| \geq 2\}$ we have

$$|u_k(z)| = \left| \frac{1}{(z^2-1)^k} \right| \leq \frac{1}{3^k} = M_k,$$

the series

$$\sum_{k=0}^{\infty} M_k = \sum_{k=0}^{\infty} \frac{1}{3^k} = \frac{3}{2}$$

is convergent.

Therefore, by the Weierstrass M-test, the series $\sum_{k=0}^{\infty} \frac{1}{(z^2-1)^k}$ converges uniformly on $\{z : |z| \geq 2\}$.

Check in Progress-II

Note: Please give solution of questions in space give below:

Q. 1 Prove that the following series converge uniformly on the sets indicated.

$\sum_{k=1}^{\infty} \frac{1}{k^2} z^k$ Converge uniformly on $\overline{D_1}(0) = \{z : |z| \le 1\}$.

Solution:

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Q. 2 Define Abel's Convergence Theorem.

Solution:

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Exercise 2. Prove that the following series converge uniformly on the sets indicated.

2 (c). $\sum_{k=0}^{\infty} \frac{z^{2k}}{z^k + 1}$ Converge uniformly on $\overline{D_r}(0) = \{z : |z| \le r\}$, where $0 < r < 1$.

Solution 2 (c).

Solution. Here $u_k(z) = \frac{1}{k^2} z^k$, and for a fixed r, with $0 < r < 1$, choose j large so that $r^j < \frac{1}{2}$ for all $k \ge j$.

Then, for all $z \in \overline{D_r}(0)$, and all $k \ge j$, we have

$$1 > 2r^j > 2r^k > 2|z^k| = |z^k| + |z^k|,$$

so that

$1-|z^k| > |z^k|$.

Recall formula (1-24) in Section 1.3.

(1-24) $|z_1+z_2| \ge |z_1|-|z_2|$.

In formula (1-24) we set $z_1 = 1$ and $z_2 = z^k$ and get

$$|z^k+1| = |1+z^k| \ge |1|-|z^k| = 1-|z^k|.$$

Now use this, together with $1-|z^k| > |z^k|$ and obtain

$$|z^k+1| \ge 1-|z^k| > |z^k|.$$

Therefore, it now follows that

$$\sum_{k=j}^{\infty}\left|\frac{z^{2k}}{z^k+1}\right| \le \sum_{k=j}^{\infty}\frac{|z^{2k}|}{|z^k+1|}$$

$$< \sum_{k=j}^{\infty}\frac{|z^{2k}|}{|z^k|}$$

$$= \sum_{k=j}^{\infty}|z^k|$$

$$\le \sum_{k=j}^{\infty}r^k$$

For $z \in \overline{D_1}(0) = \{z : |z| \le 1\}$ and all $k \ge j$, we have

$$|u_k(z)| = \left|\frac{z^{2k}}{z^k+1}\right| \le r^k = M_k.$$

The series

$$\sum_{k=j}^{\infty}M_k = \sum_{k=j}^{\infty}r^k = \frac{r^j}{1-r}$$

is known to be convergent (it is a geometric series).

Therefore, by the Weierstrass M-test, the series $\sum_{k=0}^{\infty} \frac{z^{2k}}{z^k + 1}$ converges uniformly on $\overline{D_r}(0) = \{z : |z| \le r\}$, where $0 < r < 1$.

Exercise 3. Show that $S_n(z) = \sum_{k=0}^{n-1} z^k = \frac{1 - z^n}{1 - z}$ does not converge uniformly to $f(z) = \frac{1}{1 - z}$

On the set $T = D_1(0) = \{z : |z| < 1\}$

Hint. Given $\epsilon > 0$ and a positive integer n, let $z_n = \epsilon^{\frac{1}{n}}$.

Solution 3.

Solution. Let $\epsilon = \frac{1}{2}$. Then for every positive integer n, if $z_n = \epsilon^{\frac{1}{n}} \in D_1(0)$, then

$$\begin{aligned}
|S_n(z_n) - f(z_n)| &= \left| \frac{1 - z^n}{1 - z} - \frac{1}{1 - z} \right| \\
&= \frac{|-z_n^n|}{|1 - z_n|} \\
&\ge \frac{|z_n^n|}{1 - |z_n|} \\
&= \frac{(\epsilon^{1/n})^n}{1 - \epsilon^{\frac{1}{n}}} \\
&= \frac{\epsilon}{1 - \epsilon^{\frac{1}{n}}} \\
&\ge \epsilon
\end{aligned}$$

Thus Statement is satisfied:

There exists an $\epsilon > 0$, such that for all positive integers N,
there is some $n \ge N$ and some $z_n \in T$
such that $|S_n(z_n) - f(z_n)| \ge \epsilon$.

Therefore, $S_n(z) = \sum_{k=0}^{n-1} z^k = \frac{1-z^n}{1-z}$ does not converge uniformly to $f(z) = \frac{1}{1-z}$ on $T = D_1(0) = \{z : |z| < 1\}$.

We are done.

Aside. For the real function $f(x) = \frac{1}{1-x}$ on the interval $-1 < x < 1$ and the partial sums $S_n(x) = \sum_{k=0}^{n-1} x^n = \frac{1-x^n}{1-x}$,

it is known that the partial sums of $S_n(x)$ at the point $x = -1$ exhibit the behavior

$$S_n(-1) = \frac{1-(-1)^n}{1-(-1)} = \frac{1}{2}(1-(-1)^n) = \begin{cases} 1, & \text{when } n \text{ is odd,} \\ 0, & \text{when } n \text{ is even.} \end{cases}$$

Hence for the two consecutive partial sums $S_{2k}(x)$ and $S_{2k+1}(x)$ it is easy to locate points x_{2k} and x_{2k+1} near $x = -1$,

such that $S_{2k}(x_{2k}) \approx 0$ and $S_{2k+1}(x_{2k+1}) \approx 1$.

Therefore, $\{S_n(x)\}$ does not converge uniformly to $f(x) = \frac{1}{1-x}$ on the interval $-1 < x < 1$.

Exercise 4. Why can't we use the arguments of Theorem 7.2 to prove that the geometric series $\sum_{k=0}^{\infty} z^k$ converges uniformly on all of $D_1(0)$?

Solution. The crucial step in the proof of Theorem 7.2 is the statement, "Moreover, for all $z \in D_r(\alpha)$ it is clear that

$$|c_k(z-\alpha)^k| = |c_k||z-\alpha|^k = |c_k| r^k.$$

We write $\sum_{k=0}^{\infty} c_k(z-\alpha)^k = \sum_{k=0}^{\infty} z^k$ and here we have $c_k = 1$ for all n, and

$\alpha = 0$.

If we set $r = 1$, then $\sum_{k=0}^{\infty} |c_k| r^k = \sum_{k=0}^{\infty} |1| 1^k$ does not converge.

Exercise 5. Consider the function $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$, where $n^{-z} = \exp(-z \ln n)$.

5 (a). Show that $\zeta(z)$ converges uniformly on the set $H = \{z : \text{Re}[z] \geq 2\}$.

Solution. For $z \in H$, we have

$$
\begin{aligned}
|n^{-z}| &= \left| \exp(-(x + i y) \ln n) \right| \\
&= \left| \exp(-x \ln n - i y \ln n) \right| \\
&= \left| e^{-x \ln n} \right| \left| e^{-i y \ln n} \right| \\
&= e^{-x \ln n} \\
&= n^{-x}
\end{aligned}
$$

Since $z \in H$, we have $\text{Re}[z] = x \geq 2$, so that $n^{-x} \leq n^{-2} = \frac{1}{n^2}$.

Thus, for $z \in H$ we have

$$|u_k(z)| = \left| n^{-z} \right| \leq \frac{1}{n^2} = M_n,$$

the series

$$\sum_{k=1}^{\infty} M_k = \sum_{k=1}^{\infty} \frac{1}{k^2},$$

is known to be convergent (because $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is convergent when $p > 1$).

Therefore, by the Weierstrass M-test, the series $\sum_{n=1}^{\infty} n^{-z}$ converges uniformly on $H = \{z : Re[z] \geq 2\}$.

We are done.

9.3 SUMMARY

We study in this unit abel's convergence test and its examples. We study uniform convergence test and its examples .We study M-Test and its examples. We study convergent series and relation to convergence. We study Abel's Uniform convergence series. We study A series $\sum_n u_n$ is said to converge absolutely if the series $\sum_n |u_n|$ converges, where $|u_n|$ denotes the absolute value. If a series is absolutely convergent, then the sum is independent of the order in which terms are summed. Furthermore, if the series is multiplied by another absolutely convergent series, the product series will also converge absolutely.

9.4 KEYWORD

Absolutely: with no qualification, restriction, or limitation; totally.

Uniform: remaining the same in all cases and at all times; unchanging in form or character

Abel's Test: Abel's test (also known as Abel's criterion) is a method of testing for the convergence of an infinite series. The test is named after mathematician Niels Henrik Abel. ... Abel's uniform convergence test is a criterion for the uniform convergence of a series of functions dependent on parameters.

9.5 QUESTIONS FOR REVIEW

Q. 1 Consider the function $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$, where $n^{-z} = \exp(-z \ln n)$.

Then Show that $\zeta(z)$ converges uniformly on the set $H = \{z : Re[z] \geq 2\}$

.

Q. 2 Show by example that it is not necessarily the case that $\{f_n(z)\,g_n(z)\}$ converges uniformly to $f(z)\,g(z)$ on the set T.

Q. 3 suppose that the sequences of functions $\{f_n(z)\}$ and $\{g_n(z)\}$ converge uniformly on the set T.

Then Show that the sequence $\{f_n(z) + g_n(z)\}$ converges uniformly on the set T.

Q. 4 $\sum_{k=0}^{\infty} \frac{z^{2k}}{z^k + 1}$ converge uniformly on $\overline{D_r}(0) = \{z : |z| \le r\}$, where $0 < r < 1$.

Q. 5 why can't we use the arguments of Theorem 7.2 to prove that the geometric series $\sum_{k=0}^{\infty} z^k$ converges uniformly on all of $D_1(0)$?

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9.7 ANSWER TO CHECK YOUR PROGRESS

Check in Progress-I

Answer Q. 1 Check in Section 1.5

2 Check in Section 1.5

Check In Progress-II

Answer Q. 1 Check in section 2

2 Check in Section 2.2

UNIT 10 TOPIC: FACTORIZATION IN INTEGRAL FUNCTION

STRUCTURE

10.0 Objective

10.1 Introduction

10.1.1 Factorization of an Integral Function

10.1.2 Construction of an Integral Function with Given Zeros

10.2 Weierstrass Primary Factors

10.2.1 Infinite Product Theorem

10.2.2 Approximation of functions

10.2.3 Preparation theorem

10.3 Complex Integrals

10.3.1 Euler's Factorization Method

10.4 Stone–Weierstrass theorem

10.5 Summary

10.6 Keyword

10.7 Questions for review

10.8 Suggestion Reading and References

10.9 Answer to check your progress

10.0 OBJECTIVE

- Deals with Factorization theorem
- Deals with Factorization of an Integral Function
- Deals with Weierstrass factorization theorem
- Deals with approximation of a function
- Deals with complex integral

10.1 INTRODUCTION

In mathematics, **factorization** (or **factorisation**, see English spelling differences) or **factoring** consists of writing a number or another mathematical object as a product of several *factors*, usually smaller or simpler objects of the same kind. For example, 3 × 5 is a factorization of the integer 15, and $(x - 2)(x + 2)$ is a factorization of the polynomial $x^2 - 4$.

Factorization is not usually considered meaningful within number systems possessing division, such as the real or complex numbers, since any x can be trivially written as (xy)x 1/y whenever y is not zero. However, a meaningful factorization for a rational number or a rational function can be obtained by writing it in lowest terms and separately factoring its numerator and denominator.

Factorization was first considered by ancient Greek mathematicians in the case of integers. They proved the fundamental theorem of arithmetic, which asserts that every positive integer may be factored into a product of prime numbers, which cannot be further factored into integers greater than 1. Moreover, this factorization is unique up to the order of the factors. Although integer factorization is a sort of inverse to multiplication, it is much more difficult algorithmically, a fact which is exploited in the RSA cryptosystem to implement public-key cryptography.

Polynomial factorization has also been studied for centuries. In elementary algebra, factoring a polynomial reduces the problem of finding its roots to finding the roots of the factors. Polynomials with coefficients in the integers or in a field possess the unique factorization property, a version of the fundamental theorem of arithmetic with prime numbers replaced by irreducible polynomials. In particular, a univariate polynomial with complex coefficients admits a unique (up to ordering) factorization into linear polynomials: this is a version of the fundamental theorem of algebra. In this case, the factorization can be done with root-finding algorithms. The case of polynomials with integer coefficients is fundamental for computer algebra. There are efficient computer algorithms for computing (complete) factorizations within the ring of polynomials with rational number coefficients (see factorization of polynomials).

A commutative ring possessing the unique factorization property is called a unique factorization domain. There are number systems, such as certain rings of algebraic integers, which are not unique factorization domains. However, rings of algebraic integers satisfy the weaker property of Dedekind domains: ideals factor uniquely into prime ideals.

Factorization may also refer to more general decompositions of a mathematical object into the product of smaller or simpler objects. For example, every function may be factored into the composition of a surjective function with an injective function. Matrices possess many kinds of matrix factorizations. For example, every matrix has a unique LUP factorization as a product of a lower triangular matrix *L* with all diagonal entries equal to one, an upper triangular matrix *U*, and a permutation matrix *P*; this is a matrix formulation of Gaussian elimination.

10.1.1 Factorization of an Integral Function

We know that a function which is regular in every finite region of the z-plane is called an integral function or entire function. In other words, integral function is an analytic function which has no singularity except at infinity.

e.g. e^z = 1 + z + z^2 + …

The simplest integral functions are polynomials. We know that a polynomial can be uniquely expressed as the product of linear factors in the form : f(z)= f(0) − − − 1 2 n z z ... 1 z z 1 z z 1 where z1, z2,…, zn are the zeros of the polynomial. An integral function which is not a polynomial may have an infinity of zeros zn and the product π − n z z 1 taken over these zeros may be divergent. So, an integral function cannot be always factorized in the same way as a polynomial and thus we have to consider less simple factors than − n z z 1 . We observe that (a) An integral function may have no zero e.g. e z . (b) An integral function may have finite number of zeroes e.g. polynomials of finite degree. (c) An integral function may have infinite number of zeroes. e.g. sin z, cos z.

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e.g. $e^z = 1 + z + z^2/2! + \dots$
The simplest integral functions are polynomials. We know that a polynomial can be uniquely
expressed as the product of linear factors in the form:

$$f(z) = f(0)(1 - \frac{z}{z_1})(1 - \frac{z}{z_2})\ldots\ldots\ldots(1 - \frac{z}{z_n})$$

where z1, z2,…, zn are the zeros of the polynomial.
An integral function which is not a polynomial may have an infinity of zeros zn and the product
$\pi(1 - \frac{z}{z_n})$ taken over these zeros may be divergent.
So, a integral function cannot be always factorized in the same way as a polynomial and thus we
have to consider less simple factors than $(1 - \frac{z}{z_n})$. We observe that
(a) An integral function may have no zero e.g. e^z.
(b) An integral function may have finite number of zeroes e.g. polynomials of finite degree.
(c) An integral function may have infinite number of zeroes. e.g. sin z, cos z.

Theorem : The most general integral function with no zero is the form eg(z), where g(z) is
itself an integral function.

Proof : Let f(z) be an integral function with no zero, then f □(z) is also an integral function and

so is $\frac{f\prime(z)}{f(z)}$

Let F(z) = $\int_{z_0}^{z} \frac{f'(z)dz}{f(z)}$ where the integral is taken along any path from fixed point z0 to a point z.

Thus f(z) = [log f(w) zzo] = log f(z) log f(z0)

⇾ log f(z) = F(z) + log f(z0)

⇾ f(z) = exp [log f(z0) + F(z)]

= $e^{g(z)}$, where g(z) = log f(z0) + F(z) is itself an integral function.

Hence the result.

10.1.2 Construction of an Integral Function with Given Zeros

If f(z) is an integral function

with only a finite number of zeros, say z1, z1,..,zn, then the function

$$\frac{f(z)}{(z - z_1)(z - z_2) \ldots \ldots \ldots \ldots \ldots \ldots \ldots (z - z_n)}$$

is an integral function with no zeros. Also we know that the most general form of an integral

function is $e^{g(z)}$, where g(z) is an integral function. Thus, we put

$$\frac{f(z)}{(z - z_1)(z - z_2) \ldots\ldots\ldots\ldots\ldots\ldots (z - z_n)} = e^{g(z)}$$

⇾ f(z) = $(z - z_1)(z - z_2) \ldots \ldots \ldots \ldots \ldots \ldots \ldots (z - z_n)\, e^{g(z)}$

If, however, f(z) is an integral function with an infinite number of zeros, then the only limit point

of the sequence of zeros, z1, z2, …,zn,… is the point at infinity. To determine an integral

function f(z) with an infinity of zeros, we have an important theorem due to Weierstrass.

10.2 WEIERSTRASS PRIMARY FACTORS

In mathematics, and particularly in the field of complex analysis, the Weierstrass factorization theorem asserts that every entire function can be represented as a (possibly infinite) product involving its zeroes. The theorem may be viewed as an extension of the fundamental theorem of algebra, which asserts that every polynomial may be factored into linear factors, one for each root.

The theorem, which is named for Karl Weierstrass, is closely related to a second result that every sequence tending to infinity has an associated entire function with zeroes at precisely the points of that sequence.

A generalization of the theorem extends it to meromorphic functions and allows one to consider a given meromorphic function as a product of three factors: terms depending on the function's zeros and poles, and an associated non-zero holomorphic function

The expressions $E_0(z) = 1 - z$

$E_p(z) = (1 - z) \exp(z + \frac{z^2}{2} \ldots\ldots\ldots\ldots \frac{z^p}{p})$, $p \geq 1$

are called Weierstrass primary factors. Each primary factor is an integral function which has

only a simple zero at z = 1. Thus, $E_p(z/a)$ has a simple zero at z = a and no other zero.

The behavior of $E_p(z)$ as $z \to 0$, depends upon p, since for | z | < 1, we have

$$Ep(z) = \exp[\log (1 - z) + (z + \frac{z^2}{2} \ldots\ldots\ldots\ldots \frac{z^p}{p})]$$

$$= \exp [(- z - \frac{z^2}{2} \ldots\ldots\ldots\ldots - \frac{z^p}{p} - \frac{z^{p+1}}{p+1} - \frac{z^{p+2}}{p+2}) + (z + \frac{z^2}{2} \ldots\ldots\ldots\ldots \frac{z^p}{p})]$$

$$= \exp [- \frac{z^{p+1}}{p+1} - \frac{z^{p+2}}{p+2}] = \exp [- \textstyle\sum_{n=p+1}^{\infty} z^{n} /n]$$

$$\log Ep(z) = - \frac{z^{p+1}}{p+1} - \frac{z^{p+2}}{p+2}$$

Hence if K > 1 and | z | ≤ K1 , then

| log Ep(z) | ≤ | z |p+1 + | z |p+2 + …

= | z |p+1(1 + | z | + | z |2 + ….)

= |z|p+1 (1 + 1/K + 1/ K^2 + ...)

= K/K-1 |z|p+1

This inequality helps in determining the convergence of a product of primary factors. In

particular, when | z | ≤ 1/2 , then

|log Ep(z) | ≤ 2 | z |p+1 ……………………….

10.2.1 Infinite product theorem

Weierstrass' infinite product theorem [1]: For any given sequence of points in the complex plane $\mathbf{C}$,

$$0,\ldots,0,\ \alpha_1,\ \alpha_2,\ldots,$$

$$0 < |\alpha_k| \leq |\alpha_{k+1}|, \qquad k = 1, 2, \ldots; \qquad \lim_{k\to\infty} |\alpha_k| = \infty,$$

there exists an entire function with zeros at the points α_k of this sequence and only at these points. This function may be constructed as a canonical product:

$$W(z) = z^{\lambda} \prod_{k=1}^{\infty} \left(1 - \frac{z}{\alpha_k}\right) e^{P_k(z)},$$

where λ is the multiplicity of zero in the sequence (1), and

$$P_k(z) = \frac{z}{\alpha_k} + \frac{z^2}{2\alpha_k^2} + \ldots + \frac{z^{m_k}}{2\alpha_k^{m_k}}.$$

The multipliers

$$W\left(\frac{z}{\alpha_k}; m_k\right) = \left(1 - \frac{z}{\alpha_k}\right) e^{P_k(z)}$$

are called Weierstrass prime multipliers or elementary factors. The exponents m_k are chosen so as to ensure the convergence of the product (2); for instance, the choice $m_k = k$ ensures the convergence of (2) for any sequence of the form (1).

It also follows from this theorem that any entire function $f(z)$ with zeros (1) has the form

$$f(z) = e^{g}(z) W(z),$$

where $W(z)$ is the canonical product (2) and $g(z)$ is an entire function (see also Hadamard theorem on entire functions).

Weierstrass' infinite product theorem can be generalized to the case of an arbitrary domain $D \subset \mathbf{C}$: Whatever a sequence of points $\{ \alpha_k \} \subset D$ without limit points in D, there exists a holomorphic function f in D with zeros at the points α_k and only at these points.

The part of the theorem concerning the existence of an entire function with arbitrarily specified zeros may be generalized to functions of several complex variables as follows: Let each point α of the complex space $\mathbf{C}^n$, $n \geq 1$, be brought into correspondence with one of its neighborhoods U_α and with a function f_α which is holomorphic in U_α . Moreover, suppose this is done in such a way that if the intersection $U_\alpha \cap U_\beta$ of the neighborhoods of the points α, $\beta \in \mathbf{C}^n$ is non-empty, then the fraction $f_\alpha / f_\beta \neq 0$ is a holomorphic function in $U_\alpha \cap U_\beta$. Under these conditions there exists an entire function f in $\mathbf{C}^n$ such that the fraction f / f_α is a holomorphic function at every point $\alpha \in \mathbf{C}^n$. This theorem is known as Cousin's second theorem.

10.2.2 Approximation of functions

Weierstrass' theorem on the approximation of functions: For any continuous real-valued function $f(x)$ on the interval $[a, b]$ there exists a sequence of algebraic polynomials $P_0(x), P_1(x), \ldots,$ which converges uniformly on $[a, b]$ to the function $f(x)$; established by K. Weierstrass . Similar results are valid for all spaces $L_p\ [a, b]$. The Jackson theorem is a strengthening of this theorem.

The theorem is also valid for real-valued continuous 2π-periodic functions and trigonometric polynomials, e.g. for real-valued functions which are continuous on a bounded closed domain in an m-dimensional space, or for polynomials in m variables. For generalizations, see Stone–Weierstrass theorem. For the approximation of functions of a complex variable by polynomials

10.2.3 Preparation theorem

Weierstrass' preparation theorem. A theorem obtained and originally formulated by K. Weierstrass in 1860 as a preparation lemma, used in the proofs of the existence and analytic nature of the implicit function of a complex variable defined by an equation $f(z,w)=0$ whose left-hand side is a holomorphic function of two complex variables. This theorem generalizes the following important property of holomorphic functions of one complex variable to functions of several complex variables:
If $f(z)$ is a holomorphic function of z in a neighborhood of the coordinate origin with $f(0)=0$, $f(z)\not\equiv 0$, then it may be represented in the form $f(z)=z^s g(z)$, where s is the multiplicity of vanishing of $f(z)$ at the coordinate origin, $s\geq 1$, while the holomorphic function $g(z)$ is non-zero in a certain neighborhood of the origin.

The formulation of the Weierstrass preparation theorem for functions of n complex variables, $n\geq 1$. Let

$$f(z)=f(z_1,\dots,z_n)$$

Be a holomorphic function of $z=(z_1,\dots,z_n)$ in the polydisc

$$U=\{z:\ |z_i|<a_i,\ i=1,\dots,n\},$$

And let

$$f(0)=0,\qquad f(0,\dots,0,z_n)\not\equiv 0\,.$$

Then, in some polydisc

$$V=\{z:\ |z_i|<b_i\leq a_i,\ i=1,\dots,n\},$$

the function $f(z)$ can be represented in the form

$$f(z)=[z_n^s+f_1(z_1,\dots,z_{n-1})z_n^{s-1}+\dots$$

$$\dots+f_s(z_1,\dots,z_{n-1})]\,g(z),$$

Where s is the multiplicity of vanishing of the function

$$f(z_n)=f(0,\dots,0,z_n)$$

At the coordinate origin, $s \geq 1$; the functions $f_j(z_1, ..., z_{n-1})$ are holomorphic in the polydisc

$$V' = \{(z_1, ..., z_{n-1}) : |z_i| < b_i, i = 1, ..., n-1\},$$

$$f_j(0, ..., 0) = 0, \qquad j = 1, ..., s;$$

The function $g(z)$ is holomorphic and does not vanish in V. The functions $f_j(z_1, ..., z_{n-1})$, $j = 1, ..., s$ and $g(z)$ are uniquely determined by the conditions of the theorem.

If the formulation is suitably modified, the coordinate origin may be replaced by any point $a = (a_1, ..., a_n)$ of the complex space $\mathbf{C}^n$. It follows from the Weierstrass preparation theorem that for $n > 1$, as distinct from the case of one complex variable, every neighborhood of a zero of a holomorphic function contains an infinite set of other zeros of this function.

Weierstrass' preparation theorem is purely algebraic, and may be formulated for formal power series. Let $\mathbf{C}[[z_1, ..., z_n]]$ be the ring of formal power series in the variables $z_1, ..., z_n$ with coefficients in the field of complex numbers $\mathbf{C}$; let f be a series of this ring whose terms have lowest possible degree $s \geq 1$, and assume that a term of the form cz_n^s, $c \neq 0$, exists. The series f can then be represented as

$$f = (z_n^s + f_1 z_n^{s-1} + \ldots + f_s)g,$$

Where $f_1, ..., f_s$ are series in $\mathbf{C}[[z_1, ..., z_{n-1}]]$ whose constant terms are zero, and g is a series in $\mathbf{C}[[z_1, ..., z_n]]$ with non-zero constant term. The formal power series $f_1, ..., f_s$ and g are uniquely determined by f.

A meaning which is sometimes given to the theorem is the following division theorem: Let the series

$$f \in \mathbf{C}[[z_1, ..., z_n]]$$

Satisfy the conditions just specified, and let g be an arbitrary series in $\mathbf{C}[[z_1, ..., z_n]]$. Then there exists a series

$$h \in \mathbf{C}[[z_1, ..., z_n]]$$

And series

$$a_j \in \mathbf{C}[[z_1, ..., z_{n-1}]], \qquad a_j(0, ..., 0) = 0,$$

$j = 0, ..., s-1,$

Which satisfy the following equation:

$$g = hf + a_0 + a_1 z_n + \ldots + a_{s-1} z_n^{s-1}.$$

Weierstrass' preparation theorem also applies to rings of formally bounded series. It provides a method of inductive transition, e.g. from $\mathbf{C}[[z_1, ..., z_{n-1}]]$ to $\mathbf{C}[[z_1, ..., z_n]]$. It is possible to establish certain properties of the rings $\mathbf{C}[z_1, ..., z_n]$ and $\mathbf{C}[[z_1, ..., z_n]]$ in this way, such as being Noetherian and having the unique factorization property. There exists a generalization of this theorem to differentiable functions

Check in Progress-I

Note: Please give solution of questions in space give below:

Q. 1 Define Preparation Theorem.

Solution:

………………………………………………………………………………

………………………………………………………………………………

………………………………………………………………………………

Q. 2 Define factorization of integral function.

Solution:

………………………………………………………………………………

………………………………………………………………………………

………………………………………………………………………………

10.3 COMPLEX INTEGRALS

We saw how the derivative of a complex function is defined. We now turn our attention to the problem of integrating complex functions. We will find that integrals of analytic functions are well behaved and that many properties from calculus carry over to the complex case.

We introduce the integral of a complex function by defining the integral of a complex-valued function of a real variable.

Definition (Definite Integral of a Complex Integrand). Let $f(t) = u(t) + i\,v(t)$ where u (t) and v (t) are real-valued functions of the real variable t for $a \le t \le b$. Then

$$\int_a^b f(t)\,dt = \int_a^b (u(t) + i\,v(t))\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

We generally evaluate integrals of this type by finding the antiderivatives of u (t) and v (t) and evaluating the definite integrals on the right side of Equation. That is, if $U'(t) = u(t)$ and $V'(t) = v(t)$, we have

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt = U(b) - U(a) + i\,(V(b) - V(a))$$

Example. Show that $\int_0^1 (t - i)^3\,dt = -\frac{5}{4}$.

Solution. We write the integrand in terms of its real and imaginary parts, i.e., $f(t) = (t - i)^3 = t^3 - 3t + i(-3t^2 + 1)$. Here, $u(t) = t^3 - 3t$ and $v(t) = -3t^2 + 1$. The integrals of u(t) and v(t) are

$$\int_0^1 u(t)\,dt = \int_0^1 (t^3 - 3t)\,dt = \left(\frac{t^4}{4} - \frac{3t^2}{2}\right)\Bigg|_{t=0}^{t=1} = -\frac{5}{4}$$, and

$$\int_0^1 v(t)\,dt = \int_0^1 (-3t^2 + 1)\,dt = (-t^3 + t)\Big|_{t=0}^{t=1} = 0.$$

Hence, by Definition,

$$\int_0^1 (t-i)^3\,dt = \int_0^1 u(t)\,dt + i\int_0^1 v(t)\,dt$$

$$= -\frac{5}{4} + 0\,i = -\frac{5}{4}$$

Example Show that $\int_0^{\pi/2} e^{t+it}\,dt = \frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right) + \frac{i}{2}\left(e^{\frac{\pi}{2}} + 1\right)$.

Solution. We use the method suggested by Definitions

$$\int_0^{\pi/2} e^{t+it}\,dt = \int_0^{\pi/2} e^t e^{it}\,dt = \int_0^{\pi/2} e^t (\cos t + i\sin t)\,dt$$

$$= \int_0^{\pi/2} e^t \cos t\,dt + i\int_0^{\pi/2} e^t \sin t\,dt$$

We can evaluate each of the integrals via integration by parts. For example,

$$\int_0^{\pi/2} e^t \cos t\,dt = (e^t \sin t)\Big|_{t=0}^{t=\pi/2} - \int_0^{\pi/2} \sin t\, e^t\,dt$$

$$= \left(e^{\pi/2}\sin\frac{\pi}{2} - e^0 \sin 0\right) - \int_0^{\pi/2} \sin t\, e^t\,dt$$

$$= (e^{\pi/2}\,1 - 1*0) - \int_0^{\pi/2} e^t \sin t\,dt$$

$$= e^{\pi/2} - \int_0^{\pi/2} e^t \sin t\,dt$$

$$= e^{\pi/2} - (e^t(-\cos t))\Big|_{t=0}^{t=\pi/2} + \int_0^{\pi/2} (-\cos t\, e^t)\,dt$$

$$= e^{\pi/2} + (e^t \cos t)\Big|_{t=0}^{t=\pi/2} - \int_0^{\pi/2} e^t \cos t\,dt$$

$$= e^{\pi/2} + e^{\pi/2}*0 - 1*1 - \int_0^{\pi/2} e^t \cos t\,dt$$

$$= e^{\pi/2} - 1 - \int_0^{\pi/2} e^t \cos t\,dt$$

Adding $\int_0^{\pi/2} e^t \cos t\,dt$ to both sides of this equation and then dividing by 2 gives $\int_0^{\pi/2} e^t \cos t\,dt = \frac{1}{2}(e^{\pi/2} - 1)$. Likewise, $i\int_0^{\pi/2} e^t \sin t\,dt = i\,\frac{1}{2}(e^{\pi/2} + 1)$. Therefore,

$$\int_0^{\pi/2} e^{t+it}\,dt = \frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right) + \frac{i}{2}\left(e^{\frac{\pi}{2}} + 1\right)$$

Complex integrals have properties that are similar to those of real integrals. We now trace through several commonalities. Let $f(t) = u(t) + i\,v(t)$ and $g(t) = p(t) + i\,q(t)$ be continuous on $a \le t \le b$.

Using Definition, we can easily show that the integral of their sum is the sum of their integrals, that is

$$\int_a^b (f(t) + g(t))\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt.$$

If we divide the interval $a \le t \le b$ into $a \le t \le c$ and $c \le t \le b$ and integrate f (t) over these subintervals, then we get

$$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt.$$

Similarly, if $\alpha + i\beta$ denotes a complex constant, then

$$\int_a^b (c + i\,d)\,f(t)\,dt = (c + i\,d)\int_a^b f(t)\,dt.$$

If the limits of integration are reversed, then

$$\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt.$$

The integral of the product f (t) g (t) becomes

$$\int_a^b f(t)\,g(t)\,dt = \int_a^b (u(t) + i\,v(t))\,(p(t) + i\,q(t))\,dt$$

$$= \int_a^b (u(t)\,p(t) - v(t)\,q(t))\,dt + i\int_a^b (u(t)\,q(t) + v(t)\,p(t))\,dt$$

Example Let us verify property. We start by writing

$$(c + i\,d)\,f(t) = (c + i\,d)\,(u(t) + i\,v(t))$$

$$= c\,u(t) - d\,v(t) + i\,(c\,v(t) + d\,u(t))$$

Using Definition, we write the left side of Equation as

$$\int_a^b (c + i\,d)\,(u(t) + i\,v(t))\,dt$$

$$= c\int_a^b u(t)\,dt - d\int_a^b v(t)\,dt + i\,c\int_a^b v(t)\,dt + i\,d\int_a^b u(t)\,dt$$

which is equivalent to

$$\int_a^b (c + i\,d)\,(u(t) + i\,v(t))\,dt$$

$$= c\int_a^b u(t)\,dt + i\,d\int_a^b u(t)\,dt + c\,i\int_a^b v(t)\,dt + i\,d\,i\int_a^b v(t)\,dt$$

$$= (c + i\,d)\int_a^b u(t)\,dt + (c + i\,d)\,i\int_a^b v(t)\,dt$$

$$= (c + i\,d)\left(\int_a^b u(t)\,dt + i\int_a^b v(t)\,dt\right)$$

Therefore, $\int_a^b (c + i\,d)\,f(t)\,dt = (c + i\,d)\int_a^b f(t)\,dt$

10.3.1 Euler's Factorization Method

We derive the evaluations of certain integrals of Euler type involving generalized hypergeometric series. Further, we establish a theorem on extended beta function, which provides evaluation of certain integrals in terms of extended beta function and certain special polynomials. The possibility of extending some of the derived results to multivariable case is also investigated.

A factorization algorithm which works by expressing N as a quadratic form in two different ways. Then

$$N = a^2 + b^2 = c^2 + d^2, \tag{1}$$

So

$$a^2 - c^2 = d^2 - b^2 \quad (2)$$

$$(a-c)(a+c) = (d-b)(d+b). \quad (3)$$

Let k be the greatest common divisor of $a-c$ and $d-b$ so

$$a - c = k\,l \quad (4)$$

$$d - b = k\,m \quad (5)$$

$$(l, m) = 1, \quad (6)$$

(Where (l, m) denotes the greatest common divisor of l and m), and

$$l\,(a+c) = m\,(d+b). \quad (7)$$

But since $(l, m) = 1$, $m \mid a + c$ and

$$a + c = m\,n, \quad (8)$$

Which gives

$$b + d = l\,n, \quad (9)$$

So we have

$$\left[\left(\tfrac{1}{2}k\right)^2 + \left(\tfrac{1}{2}n\right)^2\right]\left(l^2 + m^2\right) = \tfrac{1}{4}\left(k^2 + n^2\right)\left(l^2 + m^2\right) \quad (10)$$

$$= \tfrac{1}{4}\left[(k\,m)^2 + (k\,l)^2 + (n\,m)^2 + (n\,l)^2\right] \quad (11)$$

$$= \tfrac{1}{4}\left[(d-b)^2 + (a-c)^2 + (a+c)^2 + (d+b)^2\right] \quad (12)$$

$$= \tfrac{1}{4}\left(2a^2 + 2b^2 + 2c^2 + 2d^2\right) \quad (13)$$

$$= \tfrac{1}{4}(2N + 2N) \quad (14)$$

$$= N.$$

Check in Progress-II

Note: Please give solution of questions in space give below:

Q. 1 Define complex integral.

Solution:

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Q. 2 Define Euler's factorization method.

Solution:

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10. 4 STONE–WEIERSTRASS THEOREM

A broad generalization of the classical Weierstrass theorem on the approximation of functions, due to M.H. Stone (1937). Let C(X) be the ring of continuous functions on a compactum X with the topology of uniform convergence, i.e. the topology generated by the norm

‖f‖=maxx∈X|f(x)|,f∈C(X),

and let C0⊆C(X) be a subring containing all constants and separating the points of X, i.e. for any two different points x1,x2∈X1,there exists a function f∈C0 for which f(x1)≠f(x2). Then [C0]=C(X), i.e. every continuous function on X is the limit of a uniformly converging sequence of functions in C0.

Uniformly convergent series of analytic functions

Weierstrass' theorem on uniformly convergent series of analytic functions: If the terms of a series

$$s(z)=\sum_{k=0}^{\infty} u_k(z),$$

Which converges uniformly on compacta inside a domain D of the complex plane $\mathbf{C}$, are analytic functions in D, then the sum $s(z)$ is an analytic function in D. Moreover, the series

$$\sum_{k=0}^{\infty} u_k^{(m)}(z)$$

Obtained by m successive term-by-term differentiations of the series (*), for any m, also converges uniformly on compacta inside D towards the derivative $s^{(m)}(z)$ of the sum of the series (*). This theorem has been generalized to series of analytic functions of several complex variables converging uniformly on compacta inside a domain D of the complex space $\mathbf{C}^n, n \geq 1$ and the series of partial derivatives of a fixed order of the terms of the series (*) converges uniformly to the respective partial derivative of the sum of the series:

$$\frac{\partial^m s(z)}{\partial z_1^{m_1} \dots \partial z_n^{m_n}} = \sum_{k=0}^{\infty} \frac{\partial^m u_k(z)}{\partial z_1^{m_1} \dots \partial z_n^{m_n}},$$

$$z = (z_1, \dots, z_n), \qquad m = m_1 + \dots + m_n .$$

Weierstrass' theorem on uniform convergence on the boundary of a domain: If the terms of a series

$$\sum_{k=0}^{\infty} u_k(z)$$

are continuous in a closed bounded domain $\overline{D}$ of the complex plane $\mathbf{C}$ and are analytic in D, then uniform convergence of this series on the boundary of the domain implies that it converges uniformly on the closed domain $\overline{D}$.

This property of series of analytic functions is also applicable to analytic and harmonic functions defined, respectively, in a domain of the complex space $\mathbf{C}^n, n \geq 1$ or in the Euclidean space $\mathbf{R}^n, n \geq 2$. As a general rule it remains valid in all situations in which the maximum-modulus principle is applicable.

The polynomial

$$z_n^s + f_1(z_1, \dots, z_{n-1}) z_n^{s-1} + \dots +$$

$+ f_s(z_1, \ldots, z_{n-1})$,

Which occurs in the Weierstrass preparation theorem, is called a Weierstrass polynomial of degree s in z_n.

The analogue of the Weierstrass preparation theorem for differentiable functions is variously known as the differentiable preparation theorem, the Malgrange preparation theorem or the Malgrange–Mather preparation theorem. Let F be a smooth real-valued function on some neighborhood of 0 in $\mathbf{R} \times \mathbf{R}^n$ and let $F(t,0) = g(t)t^k$ with $g(0) \neq 0$ and g smooth near 0 in $\mathbf{R}$. Then the Malgrange preparation theorem says that there exists a smooth function q near zero such that $(qF)(t,x) = t^k + \sum_{i=0}^{k-1} \lambda_i(x)t^i$ for suitable smooth λ_i, and the Mather division theorem says that for any smooth G near 0 in $\mathbf{R} \times \mathbf{R}^n$ there exist smooth functions q and r on $\mathbf{R} \times \mathbf{R}^n$ near 0 such that $G = qF + r$ with $r(t,x) = \sum_{i=0}^{k-1} r_i(x)t^i$. For more sophisticated versions of the differentiable preparation and division theorems, cf. [a2]–[a4].

An important application is the differentiable symmetric function theorem (differentiable Newton theorem), which says that a germ f of a symmetric differentiable function of $x_1, \ldots, x_n$ in 0 can be written as a germ of a differentiable function in the elementary symmetric functions $\sigma_1 = x_1 + \ldots + x_n$, $\sigma_n = x_1 \ldots x_n$, [a7], [a8].

There exist also p-adic analogues of the preparation and division theorems. Let k be a complete non-Archimedean normed field (cf. Norm on a field). $T_n(k) = k\langle z_1, \ldots, z_n \rangle$ is the algebra of power series $\sum a_\alpha z^\alpha$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbf{N} \cup \{0\}$, $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$, such that $|a_\alpha| \to 0$ as $|\alpha| \to \infty$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$. The norm on $T_n(k)$ is defined by $\|\sum a_\alpha z^\alpha\| = \max_\alpha |a_\alpha|$. The subring $A_n(k)$ consists of all $f \in T_n(k)$ with $\|f\| \leq 1$ and $\mathfrak{m}_n(k)$ is the ideal of all $f \in A_n(k)$ with $|f| < 1$. Let $T_n(k)$ be the residue ring $A_n(k)/\mathfrak{m}_n(k)$, and let $f \mapsto \overline{f}$ be the quotient mapping. Then $T_n(k) = \overline{k}[z_1, ..., z_n]$, where $\overline{k}$ is the residue field of k. An element $f \in T_n(k)$ with $\|f\| = 1$ is called regular in z_n of degree d if $\overline{f}$ is of the form $\overline{f} = \lambda z_n^d + \sum_{i=0}^{d-1} c_i z_n^i$ with $c_i \in \overline{k}[z_1, ..., z_{n-1}]$ and $0 \neq \lambda \in \overline{k}$. Note that $T_{n-1}(k)[z_n] = k\langle z_1, ..., z_{n-1}\rangle[z_n]$ is naturally a subalgebra of $T_n(k)$. The p-adic Weierstrass preparation and division theorem now says: i) (division) Let $F \in T_n(k)$ be regular of degree d in z_n and let $G \in T_n(k)$. Then there exist unique elements $q \in T_n(k)$ and $r_i \in T_{n-1}(k)$, $i = 0, ..., d-1$, such that $G = qF + \sum_{i=0}^{d-1} r_i z_n^i$ and, moreover, $\|G\| = \max(\|F\|, r)$, where $r = \sum_{i=0}^{d-1} r_i z_n^i$; ii) (preparation) Let $F \in T_n(k)$ be of norm 1, then there exists a k-automorphism of $T_n(k)$ such that $\sigma(F)$ is regular in z_n.

10.5 SUMMARY

In this unit we study Weierstrass preparation theorem and its proof with examples. We study Factorization of an Integral Function and its proof. We study infinite product theorem. We study Uniformly convergent series of analytic functions and its proof. We study comples integral and its properties with examples. We study Euler's factorization method.

10.6 KEYWORD

UNIFORMLY : With equal space between each or in equal amounts; evenly

MODLUS: A constant factor or ratio

FACTORIZATION: When you break a number down into smaller numbers that, multiplied together, give you that original number. When you split a number into its factors or divisors, that's *factorization*.

10.7 QUESTIONS FOR REVIEW

Q. 1 The most general integral function with no zero is the form $e^{g(z)}$, where g (z) is itself an integral function

Q. 2 If z1, z2,…,zn,… be any sequence of numbers whose only limit point is the

point at infinity, then it is possible to construct an integral function which vanishes at each of the points zn and nowhere else.

Q. 3 If f(z) is an integral function and f(0) ¹ 0, then f(z) = f(0) P(z) $e^{g(z)}$, where P(z)

is the product of primary factors and g(z) is an integral function

Q. 4 Find $\int_{C_1(0)} (\exp(z) + \cos(z))\, z^{-1}\, dz = \int_{C_1(0)} \frac{e^z + \cos z}{z}\, dz$.

Q. 5 Find $\int_{C_1(0)} z^{-3} \sinh(z^2)\, dz = \int_{C_1(0)} \frac{\sinh(z^2)}{z^3}\, dz$.

10.8 SUGGESTION READING AND REFERENCES

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10.9 ANSWER TO CHECK YOUR PROGRESS

Check In Progress-I

Answer Q. 1 Check in Section 2.3

2 Check in Section 1.2

Check In Progress-II

Answer Q. 1 Check in section 3

2 Check in Section 3.1

UNIT 11 TOPIC: CAUCHY'S INTEGRAL FUNCTION

STRUCTURE

11.0 Objective

11.1 Introduction

11.1.1 Cauchy's Integral Formula

11.2 Sokhotskii Formulas

11.3 Leibniz Integral Rule

11.3.1 Cauchy's Integral Formula for Derivatives

11.4 Fundamental Theorem of Integration

11.4.1 Indefinite and Definite Integral

11.5 Summary

11.6 Keyword

11.7 Questions for review

11.8 Suggestion Reading and References

11.9 Answer to check your Progress

11.0 OBJECTIVES

- Deals with Cauchy's Integral Formula with its statement and proof
- Deals with non-Soviet literature Plemelj formulas
- Deals with boundary properties with analytic function
- Deals with Cauchy integral formula for derivatives
- Deals with Leibniz's Integral rule

11.1 INTRODUCTION

In mathematics, **Cauchy's integral formula**, named after Augustin-Louis Cauchy, is a central statement in complex analysis. It expresses the fact that a holomorphic function defined on a disk is completely determined by its values on the boundary of the disk, and it provides integral formulas for all derivatives of a holomorphic function. Cauchy's formula shows that, in complex analysis, "differentiation is equivalent to integration": complex differentiation, like integration, behaves well under uniform limits – a result denied in real analysis.

A Cauchy integral is a definite integral of a continuous function of one real variable. Let $f(x)$ be a continuous function on an interval $[a, b]$ and let $a = x_0 < \ldots < x_{i-1} < x_i < \ldots < x_n = b$, $\Delta x_i = x_i - x_{i-1}$, $i = 1, \ldots, n$. The limit

$$\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_{i-1}) \Delta x_i$$

is called the definite integral in Cauchy's sense of $f(x)$ over $[a, b]$ and is denoted by

$$\int_a^b f(x)\, dx .$$

The Cauchy integral is a particular case of the Riemann integral.

11.2 CAUCHY'S INTEGRAL FORMULA

A Cauchy integral is an integral with the Cauchy kernel,

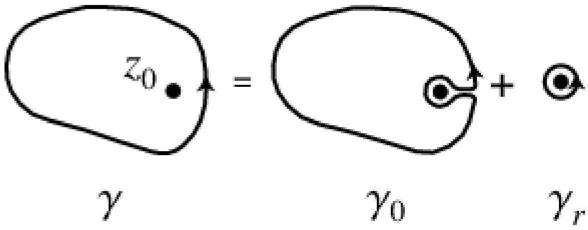
$$\frac{1}{2\pi i(\zeta - z)},$$

expressing the values of a regular analytic function $f(z)$ in the interior of a contour L in terms of its values on L. More precisely: Let $f(z)$ be a regular analytic function of the complex variable z in a domain D and let L be a closed piecewise-smooth Jordan curve lying in D together with its interior G; it is assumed that L is described in the counter-clockwise sense. Then one has the following formula, which is of fundamental importance in the theory of analytic functions of one complex variable and which is known as the Cauchy integral formula:

$$f(z) = \frac{1}{2\pi i}\int_L \frac{f(\zeta)\,d\zeta}{\zeta - z}.$$

The integral on the right of (1) is also called a Cauchy integral.

Apparently, the Cauchy integral first appeared, in certain special cases, in the work of A.L. Cauchy [1].



Cauchy's integral formula states that

$$f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)\,dz}{z - z_0}, \qquad (1)$$

Where the integral is a contour integral along the contour γ enclosing the point z_0.

It can be derived by considering the contour integral

$$\oint_\gamma \frac{f(z)\,dz}{z - z_0}, \qquad (2)$$

defining a path γ_r as an infinitesimal counterclockwise circle around the point z_0, and defining the path γ_0 as an arbitrary loop with a cut line (on which the forward and reverse contributions cancel each other out) so as to go around z_0. The total path is then

$$\gamma = \gamma_0 + \gamma_r, \qquad (3)$$

so

$$\oint_{\gamma} \frac{f(z)\, dz}{z - z_0} = \oint_{\gamma_0} \frac{f(z)\, dz}{z - z_0} + \oint_{\gamma_r} \frac{f(z)\, dz}{z - z_0}. \tag{4}$$

From the Cauchy integral theorem, the contour integral along any path not enclosing a pole is 0. Therefore, the first term in the above equation is 0 since γ_0 does not enclose the pole, and we are left with

$$\oint_{\gamma} \frac{f(z)\, dz}{z - z_0} = \oint_{\gamma_r} \frac{f(z)\, dz}{z - z_0}. \tag{5}$$

Now, let $z \equiv z_0 + r e^{i\theta}$, so $dz = i r e^{i\theta} d\theta$. Then

$$\oint_{\gamma} \frac{f(z)\, dz}{z - z_0} = \oint_{\gamma_r} \frac{f\left(z_0 + r e^{i\theta}\right)}{r e^{i\theta}}\, i r e^{i\theta}\, d\theta \tag{6}$$

$$= \oint_{\gamma_r} f\left(z_0 + r e^{i\theta}\right) i\, d\theta. \tag{7}$$

But we are free to allow the radius r to shrink to 0, so

$$\oint_{\gamma} \frac{f(z)\, dz}{z - z_0} = \lim_{r \to 0} \oint_{\gamma_r} f\left(z_0 + r e^{i\theta}\right) i\, d\theta \tag{8}$$

$$= \oint_{\gamma_r} f(z_0)\, i\, d\theta \tag{9}$$

$$= i f(z_0) \oint_{\gamma_r} d\theta \tag{10}$$

$$= 2\pi i f(z_0), \tag{11}$$

Giving (1).

If multiple loops are made around the point z_0, then equation (11) becomes

$$n(\gamma, z_0) f(z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)\, dz}{z - z_0}, \tag{12}$$

Where $n(\gamma, z_0)$ is the contour winding number?

A similar formula holds for the derivatives of $f(z)$,

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} \tag{13}$$

$$= \lim_{h\to 0} \frac{1}{2\pi i h}\left[\oint_\gamma \frac{f(z)\,dz}{z-z_0-h} - \oint_\gamma \frac{f(z)\,dz}{z-z_0}\right] \tag{14}$$

$$= \lim_{h\to 0} \frac{1}{2\pi i h} \oint_\gamma \frac{f(z)\left[(z-z_0)-(z-z_0-h)\right]dz}{(z-z_0-h)(z-z_0)} \tag{15}$$

$$= \lim_{h\to 0} \frac{1}{2\pi i h} \oint_\gamma \frac{h\,f(z)\,dz}{(z-z_0-h)(z-z_0)} \tag{16}$$

$$= \frac{1}{2\pi i} \oint_\gamma \frac{f(z)\,dz}{(z-z_0)^2}. \tag{17}$$

Iterating again,

$$f''(z_0) = \frac{2}{2\pi i} \oint_\gamma \frac{f(z)\,dz}{(z-z_0)^3}. \tag{18}$$

Continuing the process and adding the contour winding number n,

$$n(\gamma, z_0)\, f^{(r)}(z_0) = \frac{r!}{2\pi i} \oint_\gamma \frac{f(z)\,dz}{(z-z_0)^{r+1}}.$$

Cauchy integrals are thus characterized by two conditions: 1) they are evaluated along a closed, smooth (or, at least, piecewise-smooth) curve L ; and 2) their integrands have the form

$$\frac{f(\zeta)}{2\pi i(\zeta - z)},$$

Where $\zeta \in L$ and $f(z)$ is a regular analytic function on L and in the interior of L. If $z \in C\overline{G}$ (the complement to G) in the Cauchy integral, i.e. if z lies outside L, then, provided that the conditions 1) and 2) remain valid,

$$\frac{1}{2\pi i}\int_L \frac{f(\zeta)\,d\zeta}{\zeta - z} = 0, \qquad z \in C\overline{G}.$$

In particular, if L is the circle of radius ρ centered at a point z, i.e.

$$L = \left\{\zeta = z + \rho e^{i\theta} : 0 \le \theta < 2\pi\right\},$$

then (1) implies that

$$f(z)=\frac{1}{2\pi i}\int_{0}^{2\pi} f(z+\rho e^{i\theta})d\theta,$$

i.e. the value of $f(z)$ at any point $z\in D$ is equal to the arithmetic average of its values on any sufficiently small circle $L\subset D$ centered at z. Formula (1) enables one to prove all other elementary properties of analytic functions.

On the other hand, if $f(z)$ is a regular analytic function in the infinite domain $C\overline{G}$ (the exterior of the closed curve L) and on L, and if one defines

$$f(\infty)=\lim_{z\to\infty} f(z),$$

then the following formula, known as the Cauchy integral formula for an infinite domain, is valid:

$$\frac{1}{2\pi i}\int_{L}\frac{f(\zeta)d\zeta}{\zeta-z}=\begin{cases} f(\infty)-f(z), & z\in C\overline{G},\\ f(\infty), & z\in G.\end{cases}$$

Now let Γ be some (not necessarily closed) piecewise-smooth curve in the finite plane, $z\neq\infty$, let $\phi(\zeta)$ be a continuous complex function on Γ and let z be a point not on Γ. The term integral of Cauchy type is applied to the following generalization of the Cauchy integral:

$$F(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{\phi(\zeta)d\zeta}{\zeta-z},\qquad z\notin\Gamma\ .$$

The function $\phi(\zeta)$ is called the density of the integral of Cauchy type. Elementary properties of integrals of Cauchy type are:

1) $F(z)$ is a regular analytic function of z in any domain not containing points of Γ.

2) The derivatives $F^{(n)}(z)$ are given by the formulas

$$F^{(n)}(z)=\frac{n!}{2\pi i}\int_{\Gamma}\frac{\phi(\zeta)d\zeta}{(\zeta-z)^{n+1}},\qquad z\notin\Gamma;\qquad n=0,1,\ .\ .\ .\ .$$

3) $F(z)$ is regular at infinity, with $F(\infty)=0$
, $F(z)=O(1/z)$ as $z\to\infty$.

From the point of view of the general theory of analytic functions and its applications to mechanics and physics, it is of fundamental importance to consider the existence of boundary values of an integral of Cauchy type as one approaches Γ, and to find analytic expressions for these values. The Cauchy integral (1) is equal to $f(z)$ everywhere in the interior of L and vanishes identically outside L. Therefore, when an integral of Cauchy type (3) reduces to a Cauchy integral, i.e. when the conditions 1) and 2) are satisfied, then, as L is approached from the left (i.e. from its interior), the function $F(z)$ has boundary values $F^{+}(\zeta_0)=f(\zeta_0)$, and if these values are assumed on L it is continuous from the left on L at each point $\zeta_0\in L$; as L is approached from the right (i.e. from its exterior), then $F(z)$ has boundary values zero, i.e. $F^{-}(\zeta_0)=0$, and if these values are assumed on L it is continuous from the right on L at each point $\zeta_0\in L$. Thus, for a Cauchy integral

For an integral of Cauchy type of general form the matter is somewhat more complicated. Suppose that the equation of the curve Γ is $\zeta=\zeta(s)$, where s denotes the arc length reckoned from some fixed point, let $\zeta_0=\zeta(s_0)$ be an arbitrary fixed point on Γ and let Γ_ϵ be the part of Γ that remains after the smaller of the arcs with end points $\zeta(s_0-\epsilon)$ and $\zeta(s_0+\epsilon)$ is deleted from Γ. If the limit

$$\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{\Gamma_\epsilon}\frac{\phi(\zeta)\,d\zeta}{\zeta-\zeta_0}=\frac{1}{2\pi i}\int_{\Gamma}\frac{\phi(\zeta)\,d\zeta}{\zeta-\zeta_0},\qquad \zeta_0\in\Gamma,$$

Exists and is finite, it is called a singular integral. It can be proved, for example, that a singular integral (4) exists if the curve Γ is smooth in a neighbourhood of a point ζ_0 distinct from the end points of Γ and if the density $\phi(\zeta)$ satisfies a Hölder condition

$$|\phi(\zeta')-\phi(\zeta'')|\leq C|\zeta'-\zeta''|^{\mu},\qquad \mu>0.$$

Under these conditions there also exist boundary values, and these are given by the Sokhotskii formulas:

$$F^{\pm}(\zeta_0) = \pm\frac{1}{2}\phi(\zeta_0) + \frac{1}{2\pi i}\int_{\Gamma}\frac{\phi(\zeta)\,d\zeta}{\zeta-\zeta_0}, \qquad \zeta_0 \in \Gamma,$$

and the functions $F^+(z)$ and $F^-(z)$ are continuous in a neighbourhood of $\zeta_0 \in \Gamma$ from the left and right, respectively, of Γ. In the case of a Cauchy integral, the singular integral is equal to

$$\frac{f(\zeta_0)}{2},$$

$$F^+(\zeta_0) - F^-(\zeta_0) = f(\zeta_0), \qquad F^+(\zeta_0) + F^-(\zeta_0) = f(\zeta_0).$$

An equivalent form of (5) is

$$F^+(\zeta_0) - F^-(\zeta_0) = \phi(\zeta_0),$$

$$F^+(\zeta_0) + F^-(\zeta_0) = \frac{1}{\pi i}\int_{\Gamma}\frac{\phi(\zeta)\,d\zeta}{\zeta-\zeta_0}, \qquad \zeta_0 \in \Gamma.$$

The Sokhotskii formulas (5)–(7) are of fundamental importance in the solution of boundary value problems of analytic function theory, of singular integral equations connected with integrals of Cauchy type (cf. Singular integral equation), and also in the solution of various problems in hydrodynamics, elasticity theory, etc.

Check in Progress-I

Note: Please give solution of questions in space give below:

Q. 1 Define Cauchy's Integral Formula.

Solution:

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Q. 2 State Cauchy's Integral Formula.

Solution:

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Let Γ be an arbitrary rectifiable curve of length l; for simplicity it is assumed that Γ is closed. Let $\psi = \psi(s)$ be the angle between the direction of the x-axis and the tangent to Γ at the point $\zeta = \zeta(s) \in \Gamma$, regarded as a function of the arc length s, and let $\Phi(s)$ be a complex function of s of bounded variation on $[0, l]$. The expression

$$F(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{e^{i\psi}\, d\Phi(s)}{\zeta - z}, \qquad \zeta = \zeta(s), \qquad z \notin \Gamma,$$

is called an integral of Cauchy–Stieltjes type. In other words, an integral of Cauchy–Stieltjes type is an integral of Cauchy type with respect to an arbitrary finite complex Borel measure with support on Γ. If $\Phi(s)$ is absolutely continuous, then the integral of Cauchy–Stieltjes type becomes an integral of Cauchy–Lebesgue type, often called simply an integral of Cauchy type:

$$F(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{\phi(\zeta)\, d\zeta}{\zeta - z}, \qquad z \notin \Gamma,$$

where $\phi(\zeta) = \phi[\zeta(s)] = \Phi'(s)$.

Let ζ_0 be a point of Γ at which there exists a well-defined tangent, inclined to the x-axis at an angle ψ_0; such points exist almost-everywhere on a rectifiable curve. Let z be the point on the straight line passing through ζ_0 and inclined to the normal at an angle α_0, at a distance $|z - \zeta_0| = \epsilon$, i.e. $z = \zeta_0 \pm \epsilon\, i e^{i(\psi_0 + \alpha_0)}$. The difference between the integral of Cauchy–Stieltjes type (8) and the integral over Γ_ϵ,

$$W(\zeta_0; \epsilon, \alpha_0) = \frac{1}{2\pi i}\left[\int_{\Gamma} \frac{e^{i\psi}\, d\Phi(s)}{\zeta - z} - \int_{\Gamma_\epsilon} \frac{e^{i\psi}\, d\Phi(s)}{\zeta - \zeta_0}\right],$$

is defined at all points $\zeta_0 \in \Gamma$ where the tangent is defined, i.e. almost-everywhere on Γ. An important proposition in the theory of integrals of Cauchy–Stieltjes type is Privalov's fundamental lemma: The limit

$$\lim_{\epsilon \to 0} W(\zeta_0 ; \epsilon, \alpha_0) = \pm \frac{1}{2} \Phi'(\zeta_0)$$

exists for all points $\zeta_0 \in \Gamma$, with the possible exception of a point set of measure zero on Γ, independent of α_0; the convergence is uniform in α_0 in any angle $|\alpha_0| < \pi/2 - \delta$, $\delta > 0$. If the singular integral exists almost-everywhere on Γ, then the integral of Cauchy–Stieltjes type has angular boundary values $F^{\pm}(\zeta_0)$ almost-everywhere on Γ and these satisfy the Sokhotskii formulas:

$$F^{\pm}(\zeta_0) = \pm \frac{1}{2} \Phi'(s_0) + \frac{1}{2\pi i} \int_{\Gamma} \frac{e^{i\psi} \, d\Phi(s)}{\zeta - \zeta_0}, \qquad \zeta_0 \in \Gamma .$$

The converse is also true: If an integral of Cauchy–Stieltjes type has angular boundary values from both inside and outside Γ, almost-everywhere on Γ, then the singular integral exists and formulas (10) are valid almost-everywhere on Γ. As yet (1987) there is no complete solution to the problem of finding reasonably simple necessary and sufficient conditions for the existence of boundary values for integrals of Cauchy–Stieltjes type or even for integrals of Cauchy–Lebesgue type.

In contrast to the previously considered case of an integral of Cauchy type over a smooth curve Γ, an integral of Cauchy–Stieltjes type, even when it has angular boundary values, is no longer necessarily a continuous function in a neighbourhood of $\zeta_0 \in \Gamma$ from the left or right of Γ. It is known, for example, that an integral of Cauchy–Lebesgue type (9) is continuous in the closed domain $\overline{D}$ bounded by the rectifiable contour Γ, provided one additionally assumes that the density $\phi(\zeta)$ satisfies a Lipschitz condition on Γ:

$$|\phi(\zeta') - \phi(\zeta'')| \leq C|\zeta' - \zeta''|, \qquad \zeta', \zeta'' \in \Gamma .$$

One says that an integral of Cauchy–Lebesgue type (9) becomes a Cauchy integral

$$F(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{\phi(\zeta)\, d\zeta}{\zeta - z},$$

in the sense of Lebesgue, if the angular boundary values $F^{+}(\zeta_0)$ from the inside of Γ coincide with $\phi(\zeta_0)$ almost-everywhere on Γ, i.e. $F^{-}(\zeta_0) = 0$ almost-everywhere on Γ. In this context the Golubev–Privalov theorem holds: A summable function $\phi(\zeta)$ on Γ represents the angular boundary values of some Cauchy integral from the inside of Γ if and only if all its moments vanish:

$$\int_{\Gamma} \zeta^{n}\, \phi(\zeta)\, d\zeta = 0, \qquad n = 0, 1, \ldots.$$

If the analogous conditions

$$\int_{\Gamma} \zeta^{n}\, e^{i\psi}\, d\,\Phi(s) = 0, \qquad n = 0, 1, \ldots,$$

are satisfied, then the integral of Cauchy–Stieltjes type (8) becomes a Cauchy–Stieltjes integral:

$$F(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{e^{i\psi}\, d\,\Phi(s)}{\zeta - z},$$

i.e. the angular boundary values $F^{+}(\zeta_0)$ from the inside of Γ coincide with the derivative $\Phi'(s_0)$ almost-everywhere on Γ, or, stated differently, the angular boundary values $F^{-}(\zeta_0)$ from the outside of Γ vanish almost-everywhere on Γ. Conditions (13) immediately imply that the function $\Phi(s)$ is absolutely continuous on $[0, l]$ and, consequently, in this case the Cauchy–Stieltjes integral (14) is in fact a Cauchy–Lebesgue integral with density $\phi(\zeta) = \phi[\zeta(s)] = \Phi'(s)$. Thus, the class of functions representable by a Cauchy–Stieltjes integral is identical with the class of functions representable by a Cauchy–Lebesgue integral.

An important problem is the intrinsic characterization of classes of functions which are regular in a domain D bounded by a closed rectifiable curve Γ, and representable by a Cauchy integral (11), an integral of Cauchy–Lebesgue type (9), or an integral of Cauchy–Stieltjes type (8); concerning the most important classes $A(D)$, $B(D) = H_\infty(D)$, $H_p(D)$ and $N^*(D)$ see Boundary properties of analytic functions.

In the simplest case, when $D = \{z: |z| < 1\}$ is the unit disc and $\Gamma = \{z: |z| = 1\}$ is the unit circle, an integral of Cauchy–Stieltjes type, which in this case has the form

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\zeta \, d\Phi(\theta)}{\zeta - z}, \qquad |z| < 1, \qquad \zeta = e^{i\theta},$$

always represents a function of class H_p, $0 < p < 1$. The converse is false: The set of functions of classes $H_p, 0 < p < 1$ is more extensive than the set of functions representable in the form (15). On the other hand, the set of functions representable in D by a Cauchy–Stieltjes or a Cauchy integral is identical with the class H_1.

In the case of an arbitrary simply-connected domain D bounded by a rectifiable curve Γ, the class of functions representable in D by a Cauchy–Stieltjes or a Cauchy integral is identical with the Smirnov class E_1 (see Boundary properties of analytic functions). The characteristics of the classes of functions representable by an integral of Cauchy–Stieltjes type or an integral of Cauchy–Lebesgue type are considerably more complicated.

Let $f(z)$ be an arbitrary (non-analytic) function of class C^1 in a finite closed domain $\overline{D}$ bounded by a piecewise-smooth Jordan curve L. The term Cauchy integral formula is sometimes applied also to the following generalization of the classical formula (1):

$$\frac{1}{2\pi i} \int_L \frac{f(\zeta)\, d\zeta}{\zeta - z} - \frac{1}{\pi} \iint_D \frac{\partial f}{\partial \overline{\zeta}} \frac{d\xi \, d\eta}{\zeta - z} =$$

$$= \begin{cases} f(z), & z \in D, \\ 0, & z \in C\overline{D}, \end{cases}$$

Where

$$\frac{\partial f}{\partial \bar{\zeta}} = \frac{1}{2}\left(\frac{\partial f}{\partial \xi} + i\frac{\partial f}{\partial \eta}\right), \qquad \zeta = \xi + i\eta .$$

This formula first appeared, apparently, in the work of D. Pompeiu (1912). It is also known as the Pompeiu formula, the Borel–Pompeiu formula, or the Cauchy–Green formula, and is widely applied in the theory of generalized analytic functions, singular integral equations and various applied problems.

Let $f(z)$ be a regular analytic function of several complex variables $z=(z_1, \dots, z_n)$ in a closed polydisc $\overline{D}$, $D=\{z\in\mathbf{C}^n : |z_\nu - a_\nu| < r_\nu\}$. Then, at each point of D, $f(z)$ is representable by a multiple Cauchy integral:

$$f(z) = \frac{1}{(2\pi i)^n}\int_T \frac{f(\zeta)\,d\zeta}{\zeta - z},$$

where $T=\{\zeta \in \mathbf{C}^n : |\zeta_\nu - a_\nu| = r_\nu, \ \square\ \nu = 1, \dots, n\}$ is the distinguished boundary of the polydisc, $\zeta = (\zeta_1, \dots, \zeta_n)$, $d\zeta = d\zeta_1 \dots d\zeta_n$, $\zeta - z = (\zeta_1 - z_1)\dots(\zeta_n - z_n)$. Formula (17) yields a simple analogue of the Cauchy integral for a circle $L=\{z\in\mathbf{C} : |z-a| = r\}$, but when $n>1$ the integration in (17) extends not over the entire boundary of the polydisc but only over its distinguished boundary. In general, let $D = D_1 \times \dots \times D_n$ be a polycircular domain in $\mathbf{C}^n$ — a product of simply-connected plane domains D_ν with smooth boundaries $\partial D_\nu = \{z_\nu = z_\nu(t_\nu) : 0 \le t_\nu \le 1\}$; let $T = \partial D_1 \times \dots \times \partial D_n$ be the distinguished boundary of D, which is a smooth n-dimensional manifold. Formula (17) also generalizes to this case.

More profound generalizations of the Cauchy integral formula are extremely important in the theory of analytic functions of several complex variables; such generalizations are the Leray formula (which J. Leray himself called the Cauchy–Fantappié formula) and the Bochner–Martinelli representation formula. In this connection, when $n>1$ the theory is concerned mainly with boundary properties of integral representation.

11.2 SOKHOTSKII FORMULAS

In the non-Soviet literature Plemelj formulas is the usual name for what is here called Sokhotskii formulas.

Mapping properties of the singular integral operator associated to integrals of Cauchy type form an important subject. Let Γ be the graph of a Lipschitz function $\phi(x)$. The principal result, due to A.P. Calderón and in full generality to G. David, is that the singular integral operator

$$f \to \int_{\Gamma} \frac{f(\zeta)}{z-\zeta} d\zeta,$$

at first defined as a principal value integral for compactly supported smooth functions f on Γ, extends to a bounded linear operator sending $L_2(\Gamma)$ to itself, and (hence) also sending $L_p(\Gamma)$ to itself $(1 < p < \infty)$ and $L_\infty(\Gamma)$ to BMO, the functions of bounded mean oscillation.

Formally one can write:

$$\int_{\Gamma} \frac{f(\zeta)}{z-\zeta} d\zeta = \int_{\mathbf{R}} \frac{f(\xi + i\phi(\xi))}{x - \xi + i(\phi(x) - \phi(\xi))} (1 + \phi'(\xi)) d\xi =$$

$$= \sum_{0}^{\infty} \int_{\mathbf{R}} \frac{f(\xi + i\phi(\xi))(1 + i\phi(\xi))}{x - \xi} \left(\frac{(-i)(\phi(x) - \phi(\xi))}{x - \xi} \right)^{n} d\xi \ .$$

The integral operators $C_n(\phi)$ with kernel

$$\frac{1}{x-\xi} \left(\frac{\phi(x) - \phi(\xi)}{x - \xi} \right)^{n}$$

Are the so-called commutators of Calderón. These are of independent interest, e.g. in the theory of partial differential equations (cf. Differential equation, partial). The operators $C_n(\phi)$ have the same mapping properties as the Cauchy integral operator, as was shown by R.R. Coifman, A. McIntosh and Y. Meyer. The best norm estimate known at this moment (1987) is that for every $\delta > 0$ there exists a $c_\delta > 0$ such that

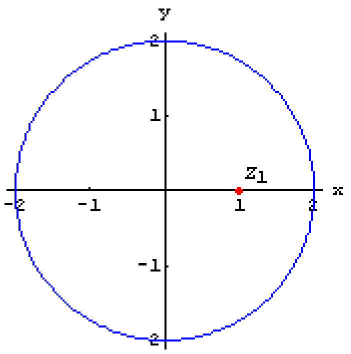
$$\|C_n(\phi)\| \leq c_\delta (n+1)^{1+\delta} \|\phi'\|_\infty^n .$$

This estimate was obtained by M. Christ and J.L. Journé.

The Cauchy integral operators as well as Calderón's commutators are examples of so-called Calderón–Zygmund operators.

For results concerning H_p functions, $0 < p < 1$, which can be represented by Cauchy integrals

Example. Show that $\int_C \frac{e^z}{z-1} dz = i 2\pi e$, where C is the circle $C: |z| = 2$ with positive orientation.

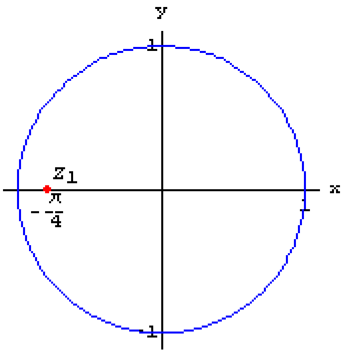


Solution. We have $f(z) = e^z$ and $f(1) = e$. The point $z_0 = 1$ lies interior to the circle, so Cauchy's integral formula implies that

$$e = f(1) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_C \frac{e^z}{z-1} dz ,$$

and multiplication by $2\pi i$ establishes the desired result.

Example . Show that $\int_C \frac{\sin(z)}{4z + \pi} dz = -i \frac{\pi\sqrt{2}}{4}$, where C is the circle $C: |z| = 1$ with positive orientation.



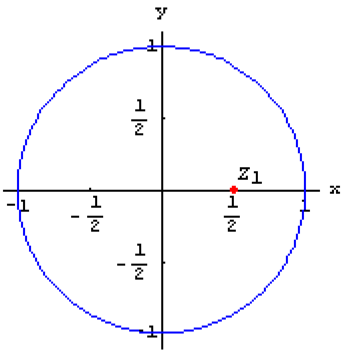
Solution. Here we have $f(z) = \sin(z)$. We manipulate the integral and use Cauchy's integral formula to obtain

$$\int_C \frac{\sin(z)}{4z+\pi}\,dz = \frac{1}{4}\int_C \frac{\sin(z)}{z+\frac{\pi}{4}}\,dz = \frac{1}{4}\int_C \frac{\sin(z)}{z-\left(-\frac{\pi}{4}\right)}\,dz$$

$$= \frac{1}{4}(2\pi i)\, f\left(-\frac{\pi}{4}\right) = \frac{1}{4}(2\pi i)\sin\left(-\frac{\pi}{4}\right)$$

$$= \frac{\pi i}{2}\left(-\frac{\sqrt{2}}{2}\right) = -i\,\frac{\pi\sqrt{2}}{4}$$

Example . Show that $\int_C \frac{e^{i\pi z}}{2z^2-5z+2}\,dz = \frac{2\pi}{3}$, where C is the circle $C: |z| = 1$ with positive orientation.



Solution. We see that $2z^2-5z+2 = (2z-1)(z-2) = 2\left(z-\frac{1}{2}\right)(z-2)$.

The only zero of this expression that lies in the interior of C is $z_0 = \frac{1}{2}$.

We set $f(z) = \frac{e^{i\pi z}}{(z-2)}$ and use Theorem 6.10 to conclude that

$$\int_C \frac{e^{i\pi z}}{2z^2 - 5z + 2}\,dz = \int_C \frac{e^{i\pi z}}{2\left(z-\frac{1}{2}\right)(z-2)}\,dz = \frac{1}{2}\int_C \frac{e^{i\pi z}}{(z-2)}\,\frac{1}{\left(z-\frac{1}{2}\right)}\,dz$$

$$= \frac{1}{2}\int_C \frac{f(z)}{\left(z-\frac{1}{2}\right)}\,dz = \frac{1}{2}(2\pi i)\,f\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}(2\pi i)\,\frac{e^{i\pi/2}}{\left(\frac{1}{2}-2\right)} = (\pi i)\,\frac{i}{\left(-\frac{3}{2}\right)}$$

$$= \frac{2\pi}{3}$$

11.3 LEIBNIZ'S INTEGRAL RULE

The Leibniz integral rule gives a formula for differentiation of a definite integral whose limits are functions of the differential variable,

$$\frac{\partial}{\partial z}\int_{a(z)}^{b(z)} f(x,z)\,dx = \int_{a(z)}^{b(z)} \frac{\partial f}{\partial z}\,dx + f(b(z),z)\,\frac{\partial b}{\partial z} - f(a(z),z)\,\frac{\partial a}{\partial z}. \qquad (1)$$

It is sometimes known as differentiation under the integral sign.

This rule can be used to evaluate certain unusual definite integrals such as

$$\phi(\alpha) = \int_0^{\pi} \ln\left(1 - 2\alpha\cos x + \alpha^2\right)dx = 2\pi\ln|\alpha| \qquad (2)$$

for $|\alpha| > 1$

Theorem (Leibniz's Rule). Let G be an open set, and let $I: a \le t \le b$ be an interval of real numbers. Let $g(z, t)$ and its partial derivative $g_z(z, t)$ with respect to z be continuous functions for all z in G and all t in I. Then

$F(z) = \int_a^b g(z, t)\,dt$ is analytic for z in G, and

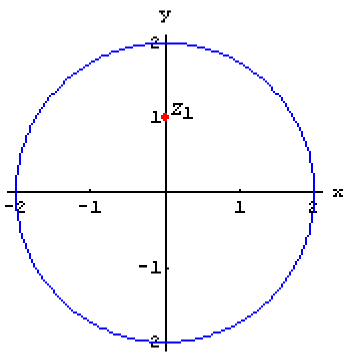
$$F'(z) = \int_a^b g_z(z, t)\, dt.$$

11.3.1 Cauchy's Integral Formulae for Derivatives

Theorem (Cauchy's Integral Formulae for Derivatives). Let $f(z)$ be analytic in the simply connected domain D, and let C be a simple closed positively oriented contour that lies in D. If z is a point that lies interior to C, then for any integer $n \geq 0$, we have

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^{n+1}}\, d\xi$$

Example . Show that $\int_C \frac{e^{z^2}}{(z - i)^4}\, dz = \frac{-4\pi}{3e}$, where C is the circle $C: |z| = 2$ with positive orientation.



Solution. If we set $f(z) = e^{z^2}$, then a straightforward calculation shows that $f^{(3)}(z) = (12z + 8z^3)e^{z^2}$. Using Cauchy's integral formulas with $n = 3$, we conclude that

$$\int_C \frac{e^{z^2}}{(z - i)^4}\, dz = \frac{2\pi i}{3!} f^{(3)}(i) = \frac{2\pi i}{3!}(12i + 8i^3)e^{i^2}$$

$$= \frac{2\pi i}{3!}(12i - 8i)e^{-1} = \frac{2\pi i}{3!}\frac{4i}{e}$$

$$= -\frac{4\pi}{3e}$$

Corollary 11.1 If $f(z)$ is analytic in the domain D, then all derivatives $f'(z), f''(z), \ldots, f^{(n)}(z), \ldots$ exists for $z \in D$ (and therefore are analytic in D).

Remark 11.1. This result is interesting, as it illustrates a big difference between real and complex functions. A real function $f(z)$ can have the property that $f'(z)$ exists everywhere in a domain D,
but $f''(z)$ exists nowhere. Corollary 11.1 states that if a complex function $f(z)$ has the property that $f'(z)$ exists everywhere in a domain D, then, remarkably, all derivatives of $f(z)$ exist in D.

11.4 THE FUNDAMENTAL THEOREMS OF INTEGRATION

Let f be analytic in the simply connected domain D. The theorems in this section show that an anti-derivative F can be constructed by contour integration. A consequence will be the fact that in a simply connected domain, the integral of an analytic function f along any contour joining z_1 to z_2 is the same, and its value is given by $F(z_2) - F(z_1)$. As a result, we can use the anti-derivative formulas from calculus to compute the value of definite integrals. The next two theorems are generalizations of the Fundamental Theorems of Calculus.

The first fundamental theorem of calculus states that,
if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{1}$$

This result, while taught early in elementary calculus courses, is actually a very deep result connecting the purely algebraic indefinite integral and the purely analytic (or geometric) definite integral.

The second fundamental theorem of calculus holds for f a continuous function on an open interval I and a any point in I, and states that if F is defined by

$$F(x) = \int_a^x f(t)\,dt, \tag{2}$$

then

$$F'(x) = f(x) \tag{3}$$

at each point in I.

The fundamental theorem of calculus along curves states that if $f(z)$ has a continuous indefinite integral $F(z)$ in a region R containing a parameterized curve $\gamma : z = z(t)$ for $\alpha \le t \le \beta$, then

$$\int_\gamma f(z)\,dz = F(z(\beta)) - F(z(\alpha)).$$

Check in Progress-II

Note : Please give solution of questions in space give below:

Q. 1 Define Fundamental Theorem For Integration.

Solution :

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Q. 2 State **Leibniz's Rule**.

Solution :

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11.4.1 Indefinite and Definite Integral

Theorem (Indefinite Integrals or Antiderivatives). Let f(z) be analytic in the simply connected domain D. If z_0 is a fixed value in D and if C is any contour in D with initial point z_0 and terminal point z, then the function

$$F(z) = \int_C f(z)\,dz = \int_{z_0}^{z} f(\xi)\,d\xi$$

is well-defined and analytic in D, with its derivative given by $F'(z) = f(z)$.

Remark . It is important to stress that the line integral of an analytic function is independent of path. In Example 6.9 we showed that $\int_{C_1} z\,dz = \int_{C_2} z\,dz = 4 + 2i$, where C_1 and C_2 were different contours joining $-1 - i$ to $3 + i$. Because the integrand $f(z) = z$ is an analytic function, Theorem 6.8 lets us know ahead of time that the value of the two integrals is the same; hence one calculation would have sufficed. If you ever have to compute a line integral of an analytic function over a difficult contour, change the contour to something easier. You are guaranteed to get the same answer. Of course, you must be sure that the function you're dealing with is analytic in a simply connected domain containing your original and new contours.

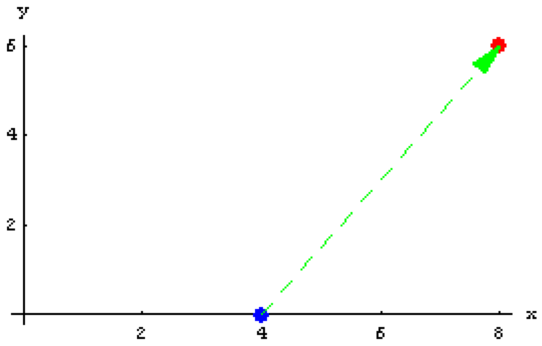
Theorem (Definite Integrals). Let f(z) be analytic in a simply connected domain D. If z_0 and z_1 are two points in D joined by a contour C, then

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0),$$

where F(z) is any antiderivative of f(z) in D.

Example 1. Show that $\int_C \frac{1}{2 z^{1/2}}\,dz = 1 + i$ where $z^{1/2}$ is the principal branch of the square root function and C is the line segment joining 4 to $8 + 6i$.

Remark. Sometimes we write this as $\int_4^{8+6i} \frac{z^{-1/2}}{2}\,dz = 1 + i$.



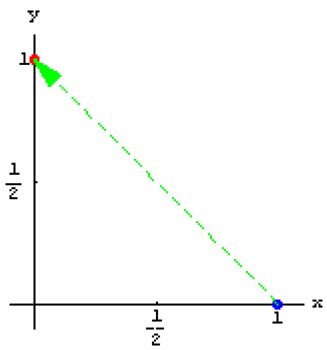
Solution. We showed that if $F(z) = z^{\frac{1}{2}}$, then $F'(z) = \frac{1}{2z^{1/2}}$, where the principal branch of the square root function is used in both the formulas for F(z) and F'(z). We note that C is contained in the simply connected domain $D_4(6+3i)$, which is the open disk of radius 4 centered at the midpoint of the segment C. Since $f(z) = \frac{1}{2z^{1/2}}$ is analytic in the domain $D_4(6+3i)$ and $F(z)$ is an anti-derivative of $f(z)$, Theorem 6.9 guarantees that

$$\int_C \frac{1}{2z^{1/2}}\,dz = \int_4^{8+6i} \frac{z^{-1/2}}{2}\,dz = F(8+6i) - F(4)$$

$$= (8+6i)^{\frac{1}{2}} - (4)^{\frac{1}{2}} = 3+i-2$$

$$= 1+i$$

Example 2 Show that $\int_C \cos(z)\,dz = -\sin 1 + i\sinh 1$, where C is the line segment between 1 and i.



Solution. An antiderivative of $f(z) = \cos z$ is $F(z) = \sin z$.

Because F(z) is entire, we use Theorem 6.9 to conclude that

$$\int_C \cos(z)\,dz = \int_1^i \cos(z)\,dz$$

$$= \int_1^i f(z)\,dz = F(i) - F(1)$$

$$= \sin i - \sin 1$$

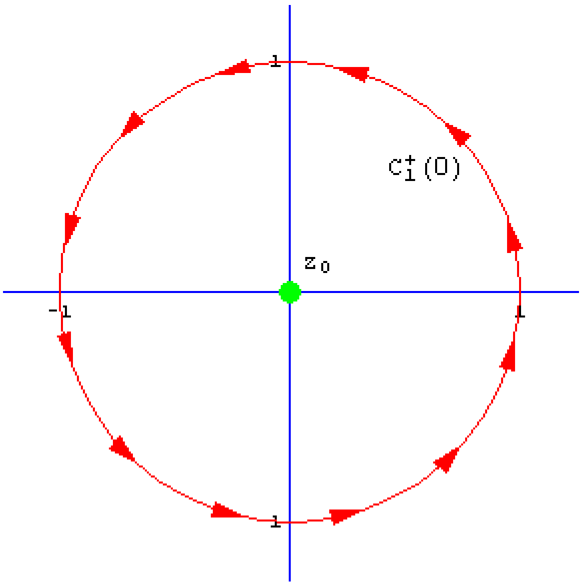
$$= i \sinh 1 - \sin 1$$

$$= -\sin 1 + i \sinh 1$$

$$= -0.841471 + 1.1752\, i$$

Example 3. Find $\int_{C_1^+(0)} z^{-n} \exp(z)\,dz = \int_{C_1^+(0)} \frac{\exp(z)}{z^n}\,dz$, where n is a positive integer.

Answer. $\int_{C_1^+(0)} z^{-n} \exp(z)\,dz = \int_{C_1^+(0)} \frac{\exp(z)}{z^{-n}}\,dz = \frac{2\pi i}{(n-1)!}$.



The point $z_0 = 0$ that lies inside the contour $C_1^+(0)$.

Solution. The integrand $\frac{\exp(z)}{z^n}$ is not defined at the point $z_0 = 0$ which lies interior to the circle $C_1^+(0)$,

and the integral $\int_{C_1^+(0)} \frac{\exp(z)}{z^n}\,dz$ has the form $\int_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz$

where $m = n - 1$,

so we can use Cauchy's Integral formula for derivatives .

Here we have $f(z) = \exp(z)$ and $f^{(n-1)}(z) = \exp(z)$ and calculation reveals that $f^{(n-1)}(z_0) = f^{(n-1)}(0) = 1$.

Applying Cauchy's Integral formula for derivatives $f^{(m)}(z_0) = \frac{m!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{m+1}}\,dz$ with $m = n - 1$ we write

$$1 = f^{(n-1)}(0) = f^{(n-1)}(z_0) = \frac{(n-1)!}{2\pi i}\int_{C_1^+(0)} \frac{f(z)}{(z-z_0)^4}\,dz = \frac{(n-1)!}{2\pi i}\int_{C_1^+(0)} \frac{\exp(z)}{(z-0)^n}\,dz = \frac{(n-1)!}{2\pi i}\int_{C_1^+(0)} \frac{\exp(z)}{z^n}\,dz$$

.

Then multiplication by $\frac{2\pi i}{3!}$ establishes the desired result

$$\int_{C_1^+(0)} \frac{\exp(z)}{z^n}\,dz = 1 * \frac{2\pi i}{(n-1)!} = \frac{2\pi i}{(n-1)!}.$$

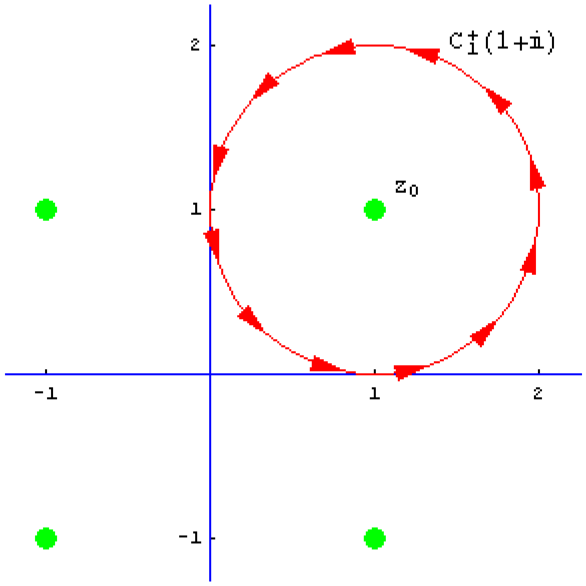
Exercise 4. Find $\int_{C_1^+(1+i)} (z^4+4)^{-1}\,dz = \int_{C_1^+(1+i)} \frac{1}{z^4+4}\,dz$

$$= \int_{C_1^+(1+i)} \frac{1}{(z+1+i)(z+1-i)(z-1+i)(z-1-i)}\,dz.$$

Answer. $\int_{C_1^+(1+i)} (z^4+4)^{-1}\,dz = \int_{C_1^+(1+i)} \frac{1}{z^4+4}\,dz =$

$$\int_{C_1^+(1+i)} \frac{1}{(z+1+i)(z+1-i)(z-1+i)(z-1-i)}\,dz = \left(\frac{1}{8} - \frac{i}{8}\right)\pi = \frac{\pi}{8} - i\frac{\pi}{8}$$

.



The point $z_0 = 1 + i$ that lies inside the contour $C_1^+(1+i)$.

Solution. This problem is a challenge because it requires us to factor $z^4 + 4$:

The linear factors of $z^4 + 4$ are

$$z^4 + 4 = (z + 1 + i)(z + 1 - i)(z - 1 + i)(z - 1 - i).$$

The integrand $\dfrac{1}{z^4+4} = \dfrac{1}{(z+1+i)(z+1-i)(z-1+i)(z-1-i)}$ is not defined at the point $z_0 = 1 + i$ which lies interior to the circle $C_1^+(1+i)$,

and the integral $\int_{C_1^+(1+i)} \dfrac{1}{(z+1+i)(z+1-i)(z-1+i)(z-1-i)}\,dz$ has the form $\int_{C_1^+(1+i)} \dfrac{f(z)}{z - z_0}\,dz$ so we can use the Cauchy Integral Formula (see Section 6.5).

Here we have $f(z) = \dfrac{1}{(z+1+i)(z+1-i)(z-1+i)}$ and calculation reveals that $f(z_0) = f(1+i) = \dfrac{1}{(2+2i)(2)(2i)} = -\dfrac{1}{16} - \dfrac{i}{16}$.

Applying the Cauchy's integral formula $f(z_0) = \dfrac{1}{2\pi i}\int_{C_1^+(1+i)} \dfrac{f(z)}{z - z_0}\,dz$

we write

$$-\frac{1}{16} - \frac{i}{16} = f(1+i) = f(z_0) = \frac{1}{2\pi i}\int_{C_1^+(1+i)} \frac{f(z)}{z - z_0}\, dz$$

$$= \frac{1}{2\pi i}\int_{C_1^+(1+i)} \frac{\frac{1}{(z+1+i)(z+1-i)(z-1+i)(z-1-i)}}{(z-1-i)}\, dz \,.$$

Then multiplication by $2\pi i$ establishes the desired result

$$\int_{C_1^+(1+i)} \frac{1}{z^4+4}\, dz = \int_{C_1^+(1+i)} \frac{1}{(z+1+i)(z+1-i)(z-1+i)(z-1-i)}\, dz$$

$$= \left(-\frac{1}{16} - \frac{i}{16}\right) * 2\pi i = \left(\frac{1}{8} - \frac{i}{8}\right)\pi = \frac{\pi}{8} - i\,\frac{\pi}{8} \,.$$

11.5 SUMMARY

We study in this unit with Cauchy's Integral Formula with its statement and proof also study non-Soviet literature Plemelj formulas and boundary properties with analytic function. We study with Cauchy integral formula for derivatives also Leibniz's Integral rule with its statement. We Study Indefinite and definite integral with its statement and examples.

11.6 KEYWORD

Indefinite: lasting for an unknown or unstated length of time

Definite: clearly stated or decided; not vague or doubtful.

Leibniz's Integral: **Leibniz** Rule. If Φ(t) = ... **integral** / x c f(z) dz is the area under the curve y = f(z), between z = a and z = x. ... Now suppose f is a function of two variables and **define**.

Boundary: a line which marks the limits of an area; a dividing line

11.7 QUESTIONS FOR REVIEW

Exercise 1. $\int_C z^2 \, dz$, where C is the line segment from $z_1 = 1 + i$ to $z_2 = 2 + i$.

Exercise 2. $\int_C \cos(z) \, dz$, where C is the line segment from $z_1 = -i$ to $z_2 = 1 + i$.

Exercise 3. $\int_C e^z \, dz$, where C is the line segment from $z_1 = 2$ to $z_2 = i \frac{\pi}{2}$.

Exercise 4. $\int_C z\, e^z \, dz$, where C is the line segment from $z_1 = -1 - \frac{i\pi}{2}$ to $z_2 = 2 + i\pi$ **Exercise 5.** Find $\int_{C_1^+(0)} z^{-4} \sin(z) \, dz = \int_{C_1^+(0)} \frac{\sin(z)}{z^4} \, dz$.

Exercise 6. Find $\int_{C_1^+(0)} (z \operatorname{Cos}(z))^{-1} \, dz = \int_{C_1^+(0)} \frac{\sec(z)}{z} \, dz$.

Exercise 7. Find $\int_{C_1^+(0)} z^{-3} \sinh(z^2) \, dz = \int_{C_1^+(0)} \frac{\sinh(z^2)}{z^3} \, dz$.

11.8 SUGGESTION READING AND REFERENCES

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i

11.9 ANSWER TO CHECK YOUR PROGRESS

Check In Progress-I

Answer Q. 1 Check in Section 1

2 Check in Section 1.2

Check In Progress-II

Answer Q. 1 Check in section 4

2 Check in Section 3

UNIT 12 TOPIC:COMPLEX EXPONENTS AND ZEROS

STRUCTURE

12.0 Objective

12.1 Introduction

12.1.1 Complex Exponents

12.2 The Rule For Exponents

12.3 Singularities, Zeros, and Poles

12.4 Zero of Order k

12.4.1 A zero of order one is sometimes called a simple zero

12.5 Summary

12.6 Keyword

12.7 Questions for review

12.8 Suggestion Reading and References

12.9 Answer to check your Progress

12.0 OBJECTIVES

- This unit deals with exponent functions
- Deals with Zeros function with its examples
- Deals with complex exponents
- Deals with rule for exponents
- Deals with Zeros of order k

12.1 INTRODUCTION

We indicated that it is possible to make sense out of expressions such as $\sqrt{1+i}$ or $(-1)^i$ without appealing to a number system beyond the framework of complex numbers. We now show how this is done by taking note of some rudimentary properties of the complex exponential and logarithm, and then using our imagination.

12.1.1 Complex Exponents

We begin by generalizing Identity. Equations show that $\log(z)$ can be expressed as the set $\log(z) = \{\mathrm{Log}(z) + i2n\pi : n \text{ is an integer}\}$. We can easily show (left as an exercise) that, for $z \neq 0$, $e^{\log_\alpha(z)} = z$,
where $\log_\alpha(z)$ is any branch of the function $\log(z)$. But this means that for any $\xi \in \log(z)$, the identity $e^\xi = z$ holds true.
Because $e^{\log(z)}$ denotes the set $\{e^\xi : \xi \in \log(z)\}$, we see that $e^{\log_\alpha(z)} = z$, for $z \neq 0$.

Next, we note that identity gives us $\log(z^n) = n\log(z)$, where n is any natural number, so that $e^{\log(z^n)} = e^{n\log(z)} = z^n$ for $z \neq 0$. With these preliminaries out of the way, we can now come up with a definition of a complex number raised to a complex power.

Definition. Let c be a complex number. We define z^c as follows

1 $z^c = e^{c\log(z)}$.

The right side of Equation is a set. This definition makes sense because, if both z and c are real numbers with $z > 0$, Equationgives the familiar (real) definition for z^c, as the following example illustrates.

Example. Use Equation 1 toevaluate $4^{1/2}$.

Solution. Calculating $4^{\frac{1}{2}} = \exp\left[\frac{1}{2}\log 4\right]$ gives

$$\frac{1}{2}\log 4 = \{\ln 2 + i n\pi : n \text{ is an integer}\}.$$

Thus $4^{\frac{1}{2}}$ is the set $\{\exp(\ln 2 + i n\pi) : n \text{ is an integer}\}$. The distinct values occur when $n = 0 \text{ and } 1$; we get $\exp(\ln 2) = 2$ and $\exp(\ln 2 + i\pi) = \exp(\ln 2)\exp(i\pi) = -2$. In other words, $4^{\frac{1}{2}} = \{-2, 2\}$.

Remark. The expression $4^{\frac{1}{2}}$ is different from $\sqrt{4}$, as the former represents the set $\{-2, 2\}$ and the latter gives only one value, $\sqrt{4} = 2$.

Because $\log(z)$ is multivalued, the function z^c will, in general, be multivalued. If we want to focus on a single value for z^c, we can do so via the function defined for $z \neq 0$ by

$$f(z) = \exp(c \operatorname{Log}(z)) \tag{2}$$

which is called the principal branch of the multivalued function z^c. Note that the principal branch of z^c is obtained from Equation 2 by replacing $\log(z)$ with the principal branch of the logarithm.

Example. Find the principal value of (**a**) $\sqrt{1+i}$, and (**b**) $(i)^i$.

Solution. From Example 5.3,

$$\operatorname{Log}(1+i) = \frac{\ln 2}{2} + i\frac{\pi}{4} = \ln 2^{\frac{1}{2}} + i\frac{\pi}{4}, \text{ and}$$

$$\operatorname{Log}(i) = i\frac{\pi}{2}.$$

Identity (2) yields the principal values of $\sqrt{1+i}$ and $(i)^i$:

$$\sqrt{1+i} = (1+i)^{\frac{1}{2}} = \exp\left[\frac{1}{2}\operatorname{Log}(1+i)\right]$$

$$= \exp\left[\frac{1}{2}\left(\ln 2^{\frac{1}{2}} + i\frac{\pi}{4}\right)\right] = \exp\left[\ln 2^{\frac{1}{4}} + i\frac{\pi}{8}\right]$$

$$= \exp\left[\ln 2^{\frac{1}{4}}\right]\exp\left[i\frac{\pi}{8}\right] = 2^{\frac{1}{4}}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$$

$$\approx 1.09868411 + 0.45508986\, i$$

and

$$(i)^i = \exp[i \operatorname{Log}(i)]$$

$$= \exp\left[i\left(i\frac{\pi}{2}\right)\right]$$

$$= \exp\left[-\frac{\pi}{2}\right]$$

$$\approx 0.20787957635$$

Note that the result of raising a complex number to a complex power may be a real number in a nontrivial way.

Let us now consider the various possibilities that may arise in the definition of z^c.

Case (i). Suppose $c = k$ where k an integer. Then, if $z = re^{i\theta} \neq 0$,

$$k \log(z) = \{k \ln(r) + i k(\theta + 2n\pi) : n \text{ is an integer}\}.$$

Recalling that the complex exponential function has period $2\pi i$, we have

$$z^k = r^k(\cos k\theta + i \sin k\theta).$$

Which is the single-valued kth power of z. This is easily verified by the computation

$$z^k = \exp[k \log z] = \exp\left[k \log\left(re^{i\theta}\right)\right]$$

$$= \exp[k \ln r + i k(\theta + 2n\pi)]$$

$$= \exp[\ln r^k + i k\theta + i 2kn\pi]$$

$$= \exp[\ln r^k] \exp[i k\theta] \exp[i 2kn\pi]$$

$$= r^k \exp(i k\theta)$$

$$= r^k(\cos k\theta + i \sin k\theta)$$

which is the single-valued kth power of z.

Case (ii). If $c = \frac{1}{k}$ where k is an integer and $z = re^{i\theta} \neq 0$, then

(3) $$\frac{1}{k}\log(z) = \left\{\frac{1}{k}\ln(r) + \frac{i(\theta + 2\pi n)}{k} : n \text{ is an integer}\right\}.$$

Hence Equation (1) becomes

$$z^{\frac{1}{k}} = r^{\frac{1}{k}}\left(\cos\frac{\theta + 2n\pi}{k} + i \sin\frac{\theta + 2n\pi}{k}\right) \text{ for } n = 0, 1, \ldots, k-1.$$

When we again use the periodicity of the complex exponential function, Equation (3) gives k distinct values corresponding to $n = 0, 1, \ldots, k-1$.

Therefore, as Example 5.6 illustrated, the fractional power $z^{\frac{1}{k}}$ is the multivalued k^{th} root function. Equation (3) is easily verified by the computation

$$z^{\frac{1}{k}} = \exp\left[\frac{1}{k}\log(z)\right] = \exp\left[\frac{1}{k}\log\left(r\,e^{i\theta}\right)\right]$$

$$= \exp\left[\frac{1}{k}(\ln r + i(\theta + 2n\pi))\right]$$

$$= \exp\left[\ln r^{\frac{1}{k}} + i\left(\frac{\theta + 2n\pi}{k}\right)\right]$$

$$= \exp\left[\ln r^{\frac{1}{k}}\right]\exp\left[i\left(\frac{\theta + 2n\pi}{k}\right)\right]$$

$$= r^{\frac{1}{k}}\left(\cos\frac{\theta + 2n\pi}{k} + i\sin\frac{\theta + 2n\pi}{k}\right)$$

Case (iii). If j and k are positive integers that have no common factors and $c = \frac{j}{k}$, then Equation (1) becomes

$$z^{\frac{j}{k}} = r^{\frac{j}{k}}\left(\cos\frac{(\theta + 2n\pi)j}{k} + i\sin\frac{(\theta + 2n\pi)j}{k}\right) \text{ for } n = 0, 1, \ldots, k-1.$$

This is easy to establish. If $z = r\,e^{i\theta}$ then

$$z^{\frac{j}{k}} = \exp\left[\frac{j}{k}\log(z)\right] = \exp\left[\frac{j}{k}\log\left(r\,e^{i\theta}\right)\right]$$

$$= \exp\left[\frac{j}{k}(\ln r + i(\theta + 2n\pi))\right]$$

$$= \exp\left[\ln r^{\frac{j}{k}} + i\frac{(\theta + 2n\pi)j}{k}\right]$$

$$= \exp\left[\ln r^{\frac{j}{k}}\right]\exp\left[i\frac{(\theta + 2n\pi)j}{k}\right]$$

$$= r^{\frac{j}{k}}\left(\cos\frac{(\theta + 2n\pi)j}{k} + i\sin\frac{(\theta + 2n\pi)j}{k}\right)$$

And again there are k distinct values corresponding to $n = 0, 1, \ldots, k-1$.

Case (IV). Suppose c is not a rational number, then there are infinitely many values for z^c, provided $z = r e^{i\theta} \neq 0$.

Example. The values of of $2^{\frac{1}{9}+\frac{i}{50}}$ are

$$\begin{aligned} 2^{\frac{1}{9}+\frac{i}{50}} &= \exp\left[\left(\frac{1}{9}+\frac{i}{50}\right)(\ln 2 + i\, 2 n \pi)\right] \\ &= \exp\left[\frac{\ln 2}{9} - \frac{n\pi}{25} + i\left(\frac{\ln 2}{50} + \frac{2 n \pi}{9}\right)\right] \\ &= \exp\left[\frac{\ln 2}{9}\right] \exp\left[-\frac{n\pi}{25}\right] \exp\left[i\left(\frac{\ln 2}{50} + \frac{2 n \pi}{9}\right)\right] \\ &= 2^{\frac{1}{9}} e^{-\frac{n\pi}{25}} \left[\cos\left(\frac{\ln 2}{50} + \frac{2 n \pi}{9}\right) + i \sin\left(\frac{\ln 2}{50} + \frac{2 n \pi}{9}\right)\right] \end{aligned}$$

where n is an integer. The principal value of $2^{\frac{1}{9}+\frac{i}{50}}$ is

$$\begin{aligned} 2^{\frac{1}{9}+\frac{i}{50}} &= 2^{\frac{1}{9}} e^{-\frac{n\pi}{25}} \left[\cos\left(\frac{\ln 2}{50}\right) + i \sin\left(\frac{\ln 2}{50}\right)\right] \\ &\approx 1.07995595696 + 0.01497232767\, i \end{aligned}$$

Figure shows the terms for this multivalued expression corresponding to $n = -9, -8, \ldots, -1, 0, 1, \ldots, 8, 9$. They exhibit a spiral pattern that is often present in complex powers.

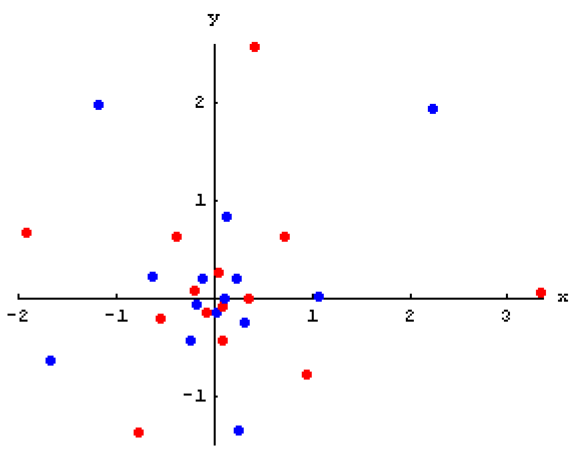

Figure Some of the "spiral pattern" of values for $2^{1/9+i/50}$.

Check in Progress-I

Note: Please give solution of questions in space give below:

Q. 1 Define Complex Exponents.

Solution:

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Q. 2 Use Equation 1 toevaluate $4^{1/2}$.

Solution:

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12.2 THE RULES FOR EXPONENTS

Some of the rules for exponents carry over from the real case. In the exercises we ask you to show that if c and d are complex numbers and $z \neq 0$, then

(5 – 24) $$z^{-c} = \frac{1}{z^c},$$

(5 – 25) $$z^c z^d = z^{c+d},$$

(5 – 26) $$\frac{z^c}{z^d} = z^{c-d},$$

(5 – 27) $$(z^c)^n = z^{cn},$$

where n is an integer.

The following example shows that Identity (1) does not hold if n is replaced with an arbitrary complex value.

Example. (a) $(i^2)^i = \exp[i \log(-1)] = \{e^{-(1+2n)\pi} : n \text{ is an integer}\}$,

and **(b)** $i^{2i} = \exp[2 i \log i] = \{e^{-(1+4n)\pi} : n \text{ is an integer}\}$.

Since these sets of solutions are not equal, Identity (1) does not always hold

We can compute the derivative of the principal branch of z^c, which is the function

$f(z) = \exp[c \operatorname{Log} z]$. By the chain rule,

$$f'(z) = \frac{c}{z} \exp[c \operatorname{Log} z].$$

If we restrict z^c to the principal branch, $z^c = \exp[c \operatorname{Log} z]$ then Equation (4) can be written in the familiar form that you learned in calculus. That is, for $z \neq 0$ and z not a negative real number,

$$\frac{d}{dz} z^c = z^c \frac{c}{z} = c z^{c-1}$$

We can use Identity (1) to define the exponential function with base b, where $b \neq 0$ is a complex number:

$b^z = \exp[z \log(b)]$.

If we specify a branch of the logarithm, then b^z will be single-valued and we can use the rules of differentiation to show that the resulting branch of b^z is an analytic function. The derivative of b^z is then given by the familiar rule

(5) $$\frac{d}{dz} b^z = b^z \log_\alpha(b)$$

where $\log_\alpha(z)$ is any branch of the logarithm whose branch cut does not include the point $z = b$.

12.3 SINGULARITIES, ZEROS, AND POLES

Recall that the point $z = \alpha$ is called a singular point, or singularity of the complex function f(z) if f is not analytic at $z = \alpha$, but every neighborhood $D_R(\alpha)$ of α contains at least one point at which f(z) is analytic. For example, the function $f(z) = \frac{1}{1 - z}$ is not analytic at $z = 1$, but is analytic for all other values of z. Thus the point $z = 1$ is a singular point of f (z). As another example, consider $g(z) = Log(z)$. We know that g (z) is analytic for all z except at the origin and at all points on the negative real-axis. Thus, the origin and each point on the negative real axis is a singularity of $g(z) = Log(z)$.

The point α is called an isolated singularity of the complex function f (z) if f is not analytic at $z = \alpha$, but there exists a real number $R > 0$ such that f (z) is analytic everywhere in the punctured disk $D_R^*(\alpha)$. The function $f(z) = \frac{1}{1 - z}$ has an isolated singularity at $z = 1$.

The function $g(z) = Log(z)$, however, the singularity at $z = 0$ (or at any point of the negative real axis) that is not isolated, because any neighborhood of contains points on the negative real axis, and $g(z) = Log(z)$ is not analytic at those points. Functions with isolated singularities have a Laurent series because the punctured disk $D_R^*(\alpha)$ is the same as the annulus $A(\alpha, 0, R)$. The logarithm function $g(z) = Log(z)$ does not have a Laurent series at any point $z = -a$ on the negative real-axis. We now look at this special case of Laurent's theorem in order to classify three types of isolated singularities.

Definition (Removable Singularity, Pole of order k, Essential Singularity). Let f (z) have an isolated singularity at α with Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - \alpha)^n \quad \text{valid for } z \in A(\alpha, 0, R).$$

Then we distinguish the following types of singularities at α.

(i) If $c_n = 0$ for $n = -1, -2, -3, \ldots$, then we say that f (z) has a removable singularity at α.

(ii) If k is a positive integer such that $c_{-k} \neq 0$ but $c_n = 0$ for $n = -k-1, -k-2, -k-3, \ldots$, then we say that f (z) has a pole of order k at α.

(iii) If $c_n \neq 0$ for infinitely many negative integers n, then we say that f(z) has an essential singularity at $z = \alpha$.

Let's investigate some examples of these three cases.

(i). If f(z) has a removable singularity at $z = \alpha$, then it has a Laurent series

$$f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$$ valid for $z \in A(\alpha, 0, R)$.

Theorem implies that the power series for f(z) defines an analytic function in the disk $D_R(\alpha)$.

If we use this series to define $f(\alpha) = c_0$, then the function f(z) becomes analytic at $z = \alpha$, removing the singularity.

For example, consider the function $f(z) = \frac{\sin(z)}{z}$. It is undefined at $z = 0$ and has an isolated singularity at $z = 0$, as the Laurent series for f(z) is

$$f(z) = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \frac{z^{11}}{11!} + \ldots\right)$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \frac{z^8}{9!} - \frac{z^{10}}{11!} + \ldots$$

valid for $|z| > 0$.

Exploration 1.

Another example is $g(z) = \dfrac{\cos(z) - 1}{z^2}$, which has an isolated singularity at the point $z = 0$, as the Laurent series for g(z) is

$$g(z) = \frac{1}{z^2}\left(-\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \ldots\right)$$

$$= -\frac{1}{2!} + \frac{z^2}{4!} - \frac{z^4}{6!} + \frac{z^6}{8!} - \frac{z^8}{10!} + \ldots$$

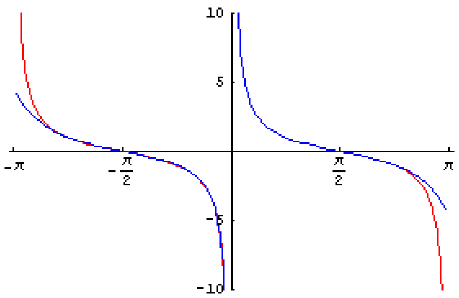
valid for $|z| > 0$. If we define $f(0) = -\frac{1}{2}$, then g(z) will be analytic for all z.

(ii). If f(z) has a pole of order k at $z = \alpha$, the Laurent series for f(z) is

$$f(z) = \sum_{n=-k}^{\infty} c_n (z-\alpha)^n$$ valid for $z \in A(\alpha, 0, R)$.

where $c_{-k} \neq 0$.

Extra Example 1. The following example will help this concept. Consider the function $f(z) = \cot z$. The leading term in the Laurent series expansion S(z) is $\frac{1}{z}$ and S(z) goes to ∞ as $z \to 0$ in the same manner as $\cot z$.



Another example is;

$$f(z) = \frac{\sin(z)}{z^3}$$

$$= \frac{1}{z^3}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \frac{z^{11}}{11!} + \ldots\right)$$

$$= \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \frac{z^6}{9!} - \frac{z^{10}}{11!} + \ldots$$

has a pole of order $k = 2$ at $z = 0$.

Exploration 2.

If f(z) has a pole of order 1 at $z = \alpha$, we say that f(z) has a simple pole at $z = \alpha$.

For example,

$$g(z) = \frac{1}{z}e^z = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{n!}z^n$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}z^{n-1}$$

$$= \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \frac{z^4}{5!} + \frac{z^5}{6!} + \ldots$$

has a simple pole at $z = 0$.

Exploration 3.

(iii). If infinitely many negative powers of $(z - \alpha)$ occur in the Laurent series, then f(z) has an essential singularity at $z = \alpha$. For example,

$$f(z) = z^2 \sin\left(\frac{1}{z}\right)$$

$$= z^2\left(\frac{1}{z} - \frac{\left(\frac{1}{z}\right)^3}{3!} + \frac{\left(\frac{1}{z}\right)^5}{5!} - \frac{\left(\frac{1}{z}\right)^7}{7!} + \frac{\left(\frac{1}{z}\right)^9}{9!} - \frac{\left(\frac{1}{z}\right)^{11}}{11!} + \ldots\right)$$

$$= z - \frac{1}{3!}z^{-1} + \frac{1}{5!}z^{3} - \frac{1}{7!}z^{-5} + \frac{1}{9!}z^{-7} - \frac{1}{11!}z^{-9} + \ldots$$

has an essential singularity at the origin.

Check in Progress-I

Note: Please give solution of questions in space give below:

Q. 1 Define Removable Singularity.

Solution:

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Q. 2 Define Pole of order k.

Solution:

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12.4 ZERO OF ORDER K

Definition (Zero of order k). A function f (z) analytic in $D_r(\alpha)$ has a zero of order k at the point $z = \alpha$ if and only if

$$f^{(n)}(\alpha) = 0 \text{ for } n = 0, 1, 2, \ldots, k-1, \text{ and } f^{(k)}(\alpha) \neq 0.$$

12.4.1 A zero of order one is sometimes called a simple zero

The word multiplicity is a general term meaning "the number of values for which a given condition holds." For example, the term is used to refer to the value of the totient valence function or the number of times a given polynomial equation has a root at a given point.

Let z_0 be a root of a function f, and let n be the least positive integer n such that $f^{(n)}(z_0) \neq 0$. Then the power series of f about z_0 begins with the $n$the term,

$$f(z) = \sum_{j=n}^{\infty} \frac{1}{j!} \frac{\partial^j f}{\partial z^j}\bigg|_{z=z_0} (z-z_0)^j,$$

And f is said to have a root of multiplicity (or "order") n. If $n = 1$, the root is called a simple root

Theorem. A function $f(z)$ analytic in $D_R(\alpha)$ has a zero of order k at the point $z = \alpha$ iff its Taylor series given by $f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$ has

$c_0 = c_1 = \ldots = c_{k-1} = 0$ and $c_k \neq 0$.

Example. We see that the function

$$f(z) = z \sin(z^2) = z^3 - \frac{z^7}{3!} + \frac{z^{11}}{5!} - \frac{z^{15}}{7!} + \ldots$$

has a zero of order $k = 3$ at $z = 0$. Definition 7.6 confirms this fact because

$$f'(z) = 2z^2 \operatorname{Cos} z^2 + \operatorname{Sin} z^2$$

$$f''(z) = 6z \operatorname{Cos} z^2 - 4z^3 \operatorname{Sin} z^2$$

$$f'''(z) = 6 \operatorname{Cos} z^2 - 8z^4 \operatorname{Cos} z^2 - 24z^2 \operatorname{Sin} z^2$$

Then, $f(0) = f''(0) = f''(0) = 0$ but $f'''(0) = 6 \neq 0$.

Theorem. Suppose f(z) is analytic in $D_R(\alpha)$. Then f(z) has a zero of order k at the point $z = \alpha$ if and only if it can be expressed in the form

(6) $f(z) = (z-\alpha)^k g(z)$,

where g(z) is analytic at $z = \alpha$ and $g(\alpha) \neq 0$.

Notes

Corollary. If f(z) and g(z) are analytic at $z = \alpha$, and have zeros of orders m and n, respectively at $z = \alpha$, then their product $h(z) = f(z)\,g(z)$ has a zero of order $m + n$ at $z = \alpha$.

Example. Let $f(z) = z^3 \sin z$. Then f (z) can be factored as the product of z^3 and $\sin z$, which have zeros of orders $m = 3$ and $n = 1$, respectively, at $z = 0$.

Hence $z = 0$ is a zero of order 4 off(z).

Theorem. A function f (z) analytic in the punctured disk $D_R^*(\alpha)$ has a pole of order k at $z = \alpha$ if and only if it can be expressed in the form

(7) $$f(z) = \frac{h(z)}{(z - \alpha)^k},$$

where the function h (z) is analytic at the point $z = \alpha$ and $h(\alpha) \neq 0$.

Corollary. If f(z) is analytic and has a zero of order k at the point $z = \alpha$, then $g(z) = \frac{1}{f(z)}$ has a pole of order k at $z = \alpha$.

Solution. $f(z) = (z - \alpha)^k r(z)$, where $r(z)$ is analytic at the point α, and $r(\alpha) \neq 0$.

Then $$g(z) = \frac{1}{f(z)} = \frac{1}{(z-\alpha)^k r(z)} = \frac{h(z)}{(z-\alpha)^k},$$

where $h(z) = \frac{1}{r(z)}$ is analytic at the point α, and $h(\alpha) \neq 0$.

Now and conclude that $g(z) = \frac{1}{f(z)}$ has a pole of order k at $z = \alpha$.

Corollary If f(z) has a pole of order k at the point $z = \alpha$, then $g(z) = \frac{1}{f(z)}$ has a removable singularity at $z = \alpha$. If we define $g(\alpha) = 0$, then g(z) has a zero of order k at $z = \alpha$.

Solution. , $f(z) = \frac{h(z)}{(z-\alpha)^k}$ is analytic in the punctured disk

$D_R^*(\alpha) = \{z : 0 < |z-\alpha| < R\}$

where $h(z)$ is analytic at the point α, and $h(\alpha) \neq 0$.

Then

$$g(z) = \frac{1}{f(z)} = \frac{1}{h(z)/(z-\alpha)^k} = \frac{(z-\alpha)^k}{h(z)} = (z-\alpha)^k r(z)$$ for

$z \in D_R^*(\alpha)$,

where $r(z) = \frac{1}{h(z)}$ is analytic at the point α, and $r(\alpha) = \frac{1}{h(\alpha)} \neq 0$.

If we define $g(\alpha) = 0$, then $g(z)$ is analytic at at the point α, and has a zero of order k at $z = \alpha$.

Corollary If f (z) and g (z) have poles of orders m and n, respectively at the point $z = \alpha$, then their product $h(z) = f(z)\, g(z)$ has a pole of order $m + n$ at $z = \alpha$.

Solution. $f(z) = \frac{r(z)}{(z-\alpha)^m}$ Is analytic in the punctured disk

$D_{R_1}^*(\alpha) = \{z : 0 < |z-\alpha| < R_1\}$,

where $r(z)$ is analytic at the point α, and $r(\alpha) \neq 0$.

Also, $g(z) = \frac{s(z)}{(z-\alpha)^n}$ is analytic in the punctured disk

$D_{R_2}^*(\alpha) = \{z : 0 < |z-\alpha| < R_2\}$,

where $s(z)$ is analytic at the point α, and $s(\alpha) \neq 0$.

Then

$$f(z)\, g(z) = \frac{r(z)}{(z-\alpha)^m} \frac{s(z)}{(z-\alpha)^n} = \frac{r(z)\, s(z)}{(z-\alpha)^{m+m}} = \frac{p(z)}{(z-\alpha)^{m+m}},$$

is analytic in the punctured disk

$D_R^*(\alpha) = \{z : 0 < |z-\alpha| < R = \min\{R_1, R_2\}\}$,

where $p(z) = r(z)\, s(z)$ is analytic at the point α, and

$p(\alpha) = r(\alpha)\, s(\alpha) \neq 0$.

Now and conclude that $h(z) = f(z)\, g(z) = \frac{r(z)\, s(z)}{(z-\alpha)^{n+m}}$ has a pole of order n+m at $z = \alpha$.

Corollary Let f(z) and g(z) be analytic with zeros of orders m and n, respectively at $z = \alpha$. Then their quotient $h(z) = \frac{f(z)}{g(z)}$ has the following behavior:

(i) If $m > n$, then h(z) has a removable singularity at $z = \alpha$. If we define $h(\alpha) = 0$, then h(z) has a zero of order $m - n$ at $z = \alpha$.

(ii) If $m < n$, then h(z) has a pole of order $n - m$ at $z = \alpha$.

(iii) If $m = n$, then h(z) has a removable singularity at $z = \alpha$, and can be defined so that h(z) is analytic at $z = \alpha$, by $h(\alpha) = \lim_{z\to\alpha} h(z)$.

Solution. $f(z) = (z-\alpha)^m r(z)$, where $r(z)$ is analytic at the point α, and $r(\alpha) \neq 0$.

Also $g(z) = (z-\alpha)^n s(z)$, where $s(z)$ is analytic at the point α, and $s(\alpha) \neq 0$.

Thus $h(z) = \frac{f(z)}{g(z)} = \frac{(z-\alpha)^m r(z)}{(z-\alpha)^n s(z)} = (z-\alpha)^{m-n} r(z)\, s(z)$, is analytic in some punctured disk $D_R^*(\alpha) = \{z : 0 < |z-\alpha| < R\}$,

where $r(z)\, s(z)$ is analytic at the point α, and $r(\alpha)\, s(\alpha) \neq 0$.

For part (i) If $m > n$, then $h(z)$ has the form

$h(z) = (z-\alpha)^{m-n} r(z)\, s(z)$, for $z \in D_R^*(\alpha) = \{z : 0 < |z-\alpha| < R\}$, where $r(z)s(z)$ is analytic at the point α, and $r(\alpha)s(\alpha) \neq 0$.

We define $h(\alpha) = 0$.

Now we have

$h(z) = (z-\alpha)^{m-n} r(z)s(z)$, for $z \in D_R(\alpha) = \{z : |z-\alpha| < R\}$,

where $r(z)s(z)$ is analytic at the point α, and $r(\alpha)s(\alpha) \neq 0$.

Now and conclude that $h(z) = (z-\alpha)^{m-n} r(z)s(z)$ has a zero of order $m-n$ at $z=\alpha$.

For part (ii) If $m < n$, then $h(z)$ has the form

$h(z) = \frac{f(z)}{g(z)} = \frac{r(z)s(z)}{(z-\alpha)^{n-m}}$, for $z \in D_R^*(\alpha) = \{z : 0 < |z-\alpha| < R\}$,

where $r(z)s(z)$ is analytic at the point α, and $r(\alpha)s(\alpha) \neq 0$.

Now and conclude that $h(z) = \frac{f(z)}{g(z)} = \frac{r(z)s(z)}{(z-\alpha)^{n-m}}$ has a pole of order n-m at $z=\alpha$.

For part (iii) If $m = n$, then $h(z)$ has the form

$h(z) = r(z)s(z)$, for $z \in D_R^*(\alpha) = \{z : 0 < |z-\alpha| < R\}$,

where $r(z)s(z)$ is analytic at the point α, and $r(\alpha)s(\alpha) \neq 0$.

All we need to do is let this definition hold in the full neighborhood

$h(z) = r(z)s(z)$, for $z \in D_R(\alpha) = \{z : |z-\alpha| < R\}$,

and $h(z)$ is analytic at $z=\alpha$.

Example Locate the zeros and poles of $h(z) = \frac{\tan z}{z}$, and determine their order.

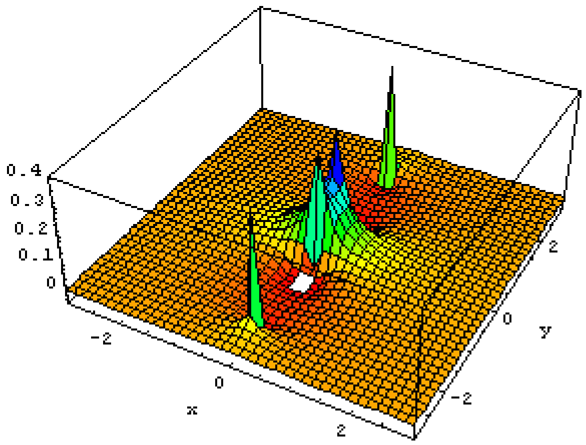
Solution. we saw that the zeros of $f(z) = \sin z$ occur at the points $z = n\pi$, where n is an integer. Because $f'(n\pi) = \cos n\pi \neq 0$, the zeros of f(z) are simple. Similarly, the function $g(z) = z\cos z$ has simple zeros at the points $z = 0$ and $z = \left(n + \frac{1}{2}\right)\pi$, where n is an integer. From the information given, we find that $h(z) = \frac{f(z)}{g(z)}$ behaves as follows:

i. h(z) has simple zeros at $z = n\pi$, where $n = \pm 1, \pm 2, \ldots$;

ii. h(z) has simple poles at $z = \left(n + \frac{1}{2}\right)\pi$, where n is an integer; and

iii. h(z) is analytic at $z = 0$ if we define $h(0) = \lim_{z \to 0} h(z) = 1$.

Example Locate the poles of $g(z) = \frac{1}{5z^4 + 26z^2 + 5}$, and specify their order.



Solution. The roots of the quadratic equation $5z^2 + 26z + 5 = 0$ occur at the points $z = -5$ and $z = -\frac{1}{5}$. If we replace z with z^2 in this equation, the function $f(z) = 5z^4 + 26z^2 + 5$ has simple zeros at the points

$z = \pm i\sqrt{5}$ and $z = \pm \frac{i}{\sqrt{5}}$. Corollary 7.5 implies that g(z) has simple poles at $z = \pm i\sqrt{5}$ and $z = \pm \frac{i}{\sqrt{5}}$.

Example . Locate the zeros and poles of $g(z) = \frac{\pi \cot(\pi z)}{z^2}$, and determine their order.

Solution. The function $f(z) = z^2 \sin \pi z$ has a zero of order $k = 3$ at $z = 0$ and simple zeros at the points $z = \pm 1, \pm 2, \ldots$. Corollary 7.5 implies that g(z) has a pole of order 3 at the point $z = 0$ and simple poles at the points $z = \pm 1, \pm 2, \ldots$.

Exercise . Let $f(z)$ be analytic and have a zero of order k at z_0. Show that the function $\frac{f'(z)}{f(z)}$ has a simple pole at z_0.

Solution. $f'(z)$ has a zero of order $k-1$ at z_0, so

$\frac{f'(z)}{f(z)}$ has a pole of order $k-(k-1) = 1$ at z_0,

which means that $\frac{f'(z)}{f(z)}$ has a simple pole at z_0.

Exercise . Let $f(z)$ have a pole of order k at z_0. Show that $f'(z)$ has a pole of order k+1 at z_0.

Solution Method I. Express f(z) in the form

$$f(z) = \frac{h(z)}{(z-z_0)^k},$$

where the function h(z) is analytic at the point $z = z_0$ and $h(z_0) \neq 0$. Then

$$f'(z) = \frac{d}{dz}\frac{h(z)}{(z-z_0)^k}$$

$$= \frac{h'(z)(z-z_0)^k - k(z-z_0)^{k-1}h(z)}{(z-z_0)^{2k}}$$

$$= \frac{h'(z)(z-z_0) - k\,h(z)}{(z-z_0)^{k+1}}$$

$$= \frac{g(z)}{(z-z_0)^{k+1}}$$

where $g(z) = h'(z)(z-z_0) - k\,h(z)$, is analytic at the point $z = z_0$,

and $h(z_0) = h'(z_0)(z_0 - z_0) - k\,h(z_0) = -k\,h(z_0) \neq 0$.

Therefore, $f'(z)$ has a pole of order k+1 at z_0.

Solution Method II. Since $f(z)$ has a pole of order k, at z_0 we can write

$$f(z) = \sum_{n=-k}^{\infty} a_n (z-z_0)^n$$

$$= \sum_{n=1}^{k} \frac{a_{-n}}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$= \frac{a_{-k}}{(z-z_0)^k} + \frac{a_{-k+1}}{(z-z_0)^{k-1}} + \ldots + \frac{a_{-1}}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

Differentiate the series termwise and obtain

$$f'(z) = \frac{d}{dz}\sum_{n=-k}^{\infty} a_n (z - z_0)^n$$

$$= \sum_{n=-k}^{\infty} \frac{d}{dz}(a_n (z - z_0)^n)$$

$$= \sum_{n=-k}^{\infty} n\, a_n (z - z_0)^{n-1}$$

$$= \sum_{n=-k-1}^{\infty} (n + 1)\, a_{n+1} (z - z_0)^n$$

$$= \sum_{n=1}^{k+1} \frac{(-n + 1)\, a_{-n+1}}{(z - z_0)^n} + \sum_{n=0}^{\infty} (n + 1)\, a_{n+1} (z - z_0)^n$$

$$= \frac{-k\, a_{-k}}{(z - z_0)^{k+1}} + \frac{(-k + 1)\, a_{-k+1}}{(z - z_0)^k} + \ldots + \frac{a_{-1}}{(z - z_0)^2} + \sum_{n=0}^{\infty} (n + 1)\, a_{n+1} (z - z_0)^n$$

Therefore, $f'(z)$ has a pole of order k+1 at z_0.

12.5 SUMMARY

In this unit we study with exponent functions in complex analysis with examples. We study h Zeros function with its examples. We study complex exponents and its properties with definition. We study singularities, poles with order k and zero of order k .We study rule for exponents. We study Zeros of order k with its examples.

12.6 KEYWORD

Exponent:a quantity representing the power to which a given number or expression is to be raised, usually expressed as a raised symbol beside the number or expression

Singularities: the state, fact, quality, or condition of being singular

Punctured: cause a sudden collapse of (mood or feeling)

12.7 QUESTIONS FOR REVIEW

Exercise 1. Locate the poles of the following functions and determine their order.

(a). $(z^2+1)^{-3}(z-1)^{-4} = \frac{1}{(z^2+1)^3(z-1)^4}$ **2 (b).**
$z^{-1}(z^2-2z+2)^{-2} = \frac{1}{z(z^2-2z+2)^2}$.

Exercise 2. Suppose that $f(z)$ has a removable singularity at z_0.

Show that the function $g(z) = \frac{1}{f(z)}$ has either a removable singularity or a pole at z_0.

Exercise 3. Let $f(z)$ be analytic and have a zero of order k at z_0. Show that $f'(z)$ has a zero of order $k-1$ at z_0.

Exercise 4. Let $f(z)$ and $g(z)$ be analytic at z_0 and have zeros of order m and n, respectively, at z_0.

What can you say about the zero of $f(z)+g(z)$ at z_0?

Exercise 5. Let $f(z)$ and $g(z)$ have poles of order m and n, respectively, at z_0.

Show that $f(z)+g(z)$ has either a pole or a removable singularity at z_0.

12.8 SUGGESTION READING AND REFERENCES

- [Ahlfors 1953/1966] L. Ahlfors, Complex Analysis, McGraw-Hill, 1953, second edition, 1966.
- [Hadamard 1893] J. Hadamard, Etude sur les Propri´et´es des Fonctions Enti`eres et en Particulier d’une ´ Fonction Consid´er´ee par Riemann, J. Math. Pures Appl. 1893, 171-215.
- [Lang 1993] S. Lang, Complex Analysis, Springer-Verlag, 1993.

- [Riemann 1859] B. Riemann, Uber die Anzahl der Primzahlen unter ener gegebenen Gr¨sse ¨ , Monats. Akad. Berlin (1859), 671-680.
- [Whittaker-Watson 1927] E.T. Whittaker, G.N. Watson, A Course of Modern Analysis, Cambridge University Press, 1927, 4th edition, 1952.

12.9 ANSWER TO CHECK YOUR PROGRESS

Check in Progress-I

Answer Q. 1 Check in Section 1.2

2 Check in Section 1.2

Check In Progress-II

Answer Q. 1 Check in section 4

2 Check in Section 4

UNIT 13 TOPIC: GENUS AND LAGUERRE'S THEOREM

STRUCTURE

13.0 Objective
13.1 Introduction
13.1.1 Hadamard's Factorization Theorem
13.1.2 Cauchy-Hadamard theorem
13.1.3 Disc Of Convergence
13.1.4 Reinhardt Domain
13.2 Genus of an Entire Function
13.2.1Entire Function
13.3 Laguerre's method
13.3.1 Laguerre's Polynomials
13.3.2 Laguerre's Transform
13.3.3 Laguerre's Function
13.3.4 Laguerre's Differential Equation
13.4 Contour Integral
13.4.1 Complex Integral
13.5 Summary
13.6 Keyword
13.7 Questions for review
13.8 Suggestion Reading and References
13.9 Answer to check your Progress

13.0 OBJECTIVE

- Deals with Genus theorem
- Deals with Hadamard's Factorization Theorem
- Deals with Cauchy-Hadamard Theorem and its proof with statement
- Deals with Genus of an Entire Function
- Deals with Entire function with its proof

- Deals with Laguerre's method
- Deals with Laguerre's Differential Equation

13.1 INTRODUCTION

Let f:C→C be a nonzero entire function of finite rank p∈Np∈N.

Let 0 be a zero of f of multiplicity m≥0.

Let ⟨an⟩⟨an⟩ be the sequence of nonzero zeroes of ff, repeated according to multiplicity.

Genus of an entire function

Converges. The non-negative integer is called the **genus** of the entire function. If the order ρ is not an integer, then is the integer part of. If the order is a positive integer.

In complex analysis, an **entire function**, also called an **integral function,** is a complex-valued function that is holomorphic at all finite points over the whole complex plane. Typical examples of entire functions are polynomials and the exponential function, and any finite sums, products and compositions of these, such as the trigonometric functions sine and cosine and their hyperbolic counterparts sinh and cosh, as well as derivatives and integrals of entire functions such as the error function. If an entire function *f*(*z*) has a root at *w*, then *f* (*z*)/(*z*−*w*), taking the limit value at *w*, is an entire function. On the other hand, neither the natural logarithm nor the square root is an entire function, nor can they be continued analytically to an entire function.

13.1.1 Hadamard Factorization Theorem

Let f be an entire function of finite order λ and $\{a_j\}$ the zeros of f, listed with multiplicity, then the rank p of f is defined as the least positive integer such that

$$\sum_{a_n \neq 0} |a_n|^{-(p+1)} < \infty. \tag{1}$$

Then the canonical Weierstrass product is given by

$$f(z) = e^{g(z)} P(z), \tag{2}$$

and g has degree $q \leq \lambda$. The genus μ of f is then defined as $\max(p, q)$, and the Hadamard factorization theory states that an entire function of finite order λ is also of finite genus μ, and

$$\mu \leq \lambda.$$

13.1.2 Cauchy-Hadamard theorem

Consider a complex power series

$$f(z) = \sum_{k=0}^{\infty} c_k (z - a)^k$$

and let

$$\Lambda = \lim_{k \to \infty} \sup |c_k|^{1/k}.$$

If $\Lambda = \infty$, then the series (1) is convergent only at the point $z = a$; if $0 < \Lambda < \infty$, then the series (1) is absolutely convergent in the disc $z - a| < R$ where

$$R = \frac{1}{\Lambda},$$

and divergent outside the disc, where $z - a| > R$; if $\Lambda = 0$, the series (1) is absolutely convergent for all $z \in \mathbf{C}$. The content of the Cauchy–Hadamard theorem is thus expressed by the Cauchy–Hadamard formula (2), which should be understood in this context in a broad sense, including $1/\infty = 0$ and $1/0 = \infty$. In other words, the Cauchy–Hadamard theorem states that the interior of the set of points at which the series (1) is (absolutely) convergent is the disc $z - a| < R$ of radius (2). In the case of a real power series (1), formula (2) defines the "radius" of the interval of convergence: $a - R < x < a + R$. Essentially, the Cauchy–Hadamard theorem was stated by A.L. Cauchy in his lectures [1] in 1821; it was J. Hadamard [2] who made the formulation and the proof fully explicit.

For power series

$$f(z)=\sum_{k_1,\dots,k_n=0}^{\infty} c_{k_1,\dots,k_n}(z_1-a_1)^{k_1}\dots(z_n-a_n)^{k_n}$$

in n complex variables $z=(z_1,\dots,z_n)$, $n\geq 1$, one has the following generalization of the Cauchy–Hadamard formula:

$$\lim_{|k|\to\infty} |c_{k_1,\dots,k_n}|^{1/|k|} r_1^{k_1}\dots r_n^{k_n}=1,$$
$$|k|=k_1+\dots+k_n,$$

which is valid for the associated radii of convergence $r_1,\dots,r_n$ of the series (3) (see Disc of convergence). Writing (4) in the form $\Phi(r_1,\dots,r_n)=0$, one obtains an equation defining the boundary of a certain logarithmically convex Reinhardt domain with centre a, which is the interior of the set of points at which the series (3) is absolutely convergent ($n>1$).

13.1.3 Disc of Convergence

Disc of Convergence of a power series

$$f(z)=\sum_{k=0}^{\infty} c_k(z-a)^k$$

The disc $\Delta=\{z:\ |z-a|<R\}$, $z\in\mathbf{C}$, in which the series is absolutely convergent, while outside the disc (for $|z-a|>R$) it is divergent. In other words, the disc of convergence Δ is the interior of the set of points of convergence of the series . Its radius R is called the radius of convergence of the series. The disc of convergence may shrink to the point a when $R=0$, and it may be the entire open plane, when $R=\infty$. The radius of convergence R is equal to the distance of the centre a to the set of singular points of $f(z)$ (for the determination of R in terms of the coefficients c_k of the series see Cauchy–Hadamard theorem). Any disc $\Delta=\{z:\ |z|<R\}$, $0\leq R\leq\infty$, in the z-plane is the disc of convergence of some power series.

For a power series

$$f(z) = f(z_1, \ldots, z_n) =$$

$$= \sum_{k_1, \ldots, k_n = 0}^{\infty} c_{k_1, \ldots, k_n} (z_1 - a_1)^{k_1} \ldots (z_n - a_n)^{k_n}$$

in several complex variables $z_1, \ldots, z_n$, $n > 1$, a polydisc of convergence of the series (2) is defined to be any polydisc

$$\Delta_n = \{z = (z_1, \ldots, z_n) : |z_\nu - a_\nu| < R_\nu,\ \nu = 1, \ldots, n\}$$

at all points of which the series (2) is absolutely convergent, while in any polydisc

$$\{z = (z_1, \ldots, z_n) : |z_\nu - a_\nu| < R'_\nu,\ \nu = 1, \ldots, n\},$$

where $R'_\nu \geq R_\nu$ and at least one of the latter inequalities is strict, there is at least one point at which the series is divergent. The radii R_ν, $\nu = 1, \ldots, n$, $0 \leq R_\nu \leq \infty$, of the polydisc of convergence are called the associated radii of convergence of the series (2). They are in a well-defined relationship with the coefficients of the series, so that any polydisc with centre a and with radii satisfying this relationship is the polydisc of convergence of a series (2) (cf. Cauchy–Hadamard theorem). Any polydisc Δ_n, $0 \leq R_\nu \leq \infty$, $\nu = 1, \ldots, n$, in the complex space $\mathbf{C}^n$ is the polydisc of convergence for some power series in n complex variables. When $n > 1$ the interior of the set of points of absolute convergence of a series (2) is more complicated — it is a logarithmically convex complete Reinhardt domain with centre a in $\mathbf{C}^n$ (cf. Reinhardt domain).

13.1.4 Reinhardt Domain

multiple-circled domain

A domain D in the complex space $\mathbf{C}^n$, $n \geq 1$, with centre at a point $a = (a_1, \ldots, a_n) \in \mathbf{C}^n$, with the following property: Together with any point $z^0 = (z_1^0, \ldots, z_n^0) \in D$, the domain also contains the set

$$\{z = (z_1, \ldots, z_n) : |z_\nu - a_\nu| = |z_\nu^0 - a_\nu|,\ \nu = 1, \ldots, n\}.$$

A Reinhardt domain D with $\alpha = 0$ is invariant under the transformations $\{z^0\} \to \{z_\nu^0 e^{i\theta_\nu}\}$, $0 \leq \theta_\nu < 2\pi$, $\nu = 1,\dots,n$. The Reinhardt domains constitute a subclass of the Hartogs domains (cf. Hartogs domain) and a subclass of the circular domains, which are defined by the following condition: Together with any $z^0 \in D$, the domain contains the set

$$\left\{ z = (z_1,\dots,z_n) : z = \alpha + (z^0 - \alpha)e^{i\theta}, 0 \leq \theta < 2\pi \right\},$$

i.e. all points of the circle with centre α and radius $|z^0 - \alpha| = (\sum_{\nu=1}^{n} |z_\nu^0 - \alpha_\nu|^2)^{1/2}$ that lie on the complex line through α and z^0.

A Reinhardt domain D is called a complete Reinhardt domain if together with any point $z^0 \in D$ it also contains the polydisc

$$\left\{ z = (z_1,\dots,z_n) : |z_\nu - \alpha_\nu| \leq |z_\nu^0 - \alpha_\nu|,\ \nu = 1,\dots,n \right\}.$$

A complete Reinhardt domain is star-like with respect to its centre α (cf. Star-like domain).

Examples of complete Reinhardt domains are balls and polydiscs in $\mathbf{C}^n$. A circular domain D is called a complete circular domain if together with any pont $z^0 \in D$ it also contains the entire disc $\left\{ z = \alpha + (z^0 - \alpha)\zeta : |\zeta| \leq 1 \right\}$.

A Reinhardt domain D is called logarithmically convex if the image $\lambda(D^*)$ of the set

$$D^* = \{ z = (z_1,\dots,z_n) \in D : z_1 \dots z_n \neq 0 \}$$

under the mapping

$$\lambda : z \to \lambda(z) = (\ln |z_1|,\dots,\ln |z_n|)$$

is a convex set in the real space $\mathbf{R}^n$. An important property of logarithmically-convex Reinhardt domains is the following: Every such domain in $\mathbf{C}^n$ is the interior of the set of points of absolute convergence (i.e. the domain of convergence) of some power series in $z_1 - \alpha_1,\dots,z_n - \alpha_n$, and conversely: The domain of convergence of any power series in $z_1, \ldots, z_n$ is a logarithmically-convex Reinhardt domain with centre $a = 0$.

13.2 GENUS OF AN ENTIRE FUNCTION

The integer equal to the larger of the two numbers p and q in the representation of the entire function $f(z)$ in the form

$$f(z) =$$

$$= z^{\lambda} e^{Q(z)} \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right) \exp\left(\frac{z}{a_k} + \frac{z^2}{2a_k^2} + \square \ldots + \frac{z^p}{pa_k^p}\right),$$

where q is the degree of the polynomial $Q(z)$ and p is the least integer satisfying the condition

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|^{p+1}} < \infty \ .$$

The number p is called the genus of the product appearing in formula (*).

13.2.1 Entire function

A function that is analytic in the whole complex plane (except, possibly, at the point at infinity). It can be expanded in a power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad a_k = \frac{f^{(k)}(0)}{k!}, \qquad k \geq 0,$$

which converges in the whole complex plane, $\lim_{k \to \infty} |a_k|^{1/k} = 0$.

If $f(z) \neq 0$ everywhere, then $f(z) = e^{P(z)}$, where $P(z)$ is an entire function. If there are finitely many points at which $f(z)$ vanishes and these points are $z_1, \ldots, z_k$ (they are called the zeros of the function), then

$$f(z) = (z - z_1)\ldots(z - z_k)e^{P(z)},$$

where $P(z)$ is an entire function.

In the general case when $f(z)$ has infinitely many zeros $z_1, z_2, \ldots,$ there is a product representation (see Weierstrass theorem on infinite products)

$$f(z) = z^{\lambda} e^{P(z)} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) \exp\left(\frac{z}{z_k} + \ldots + \frac{z^k}{k z_k^k}\right),$$

where $P(z)$ is an entire function, $\lambda = 0$ if $f(0) \neq 0$, and λ is the multiplicity of the zero $z = 0$ if $f(0) = 0$.

Let

$$M(r) = \max_{|z| \leq r} |f(z)| .$$

If for large r the quantity $M(r)$ grows no faster than r^{μ}, then $f(z)$ is a polynomial of degree not exceeding μ. Consequently, if $f(z)$ is not a polynomial, then $M(r)$ grows faster than any power of r. To estimate the growth of $M(r)$ in this case one takes as a comparison function the exponential function.

By definition, $f(z)$ is an entire function of finite order if there is a finite number μ such that

$$M(r) < e^{r^{\mu}}, \qquad r > r_0 .$$

The greatest lower bound ρ of the set of numbers μ satisfying this condition is called the order of the entire function $f(z)$. The order can be computed by the formula

$$\rho = \overline{\lim_{k \to \infty}} \frac{k \ln k}{\ln |1 / a_k|} .$$

If $f(z)$ of order ρ satisfies the condition

$$M(r) < e^{\alpha r^{\rho}}, \qquad \alpha < \infty, \qquad r > r_0,$$

then one says that $f(z)$ is a function of order ρ and of finite type. The greatest lower bound σ of the set of numbers α satisfying this condition is called the type of the entire function $f(z)$. It is determined by the formula

$$\overline{\lim_{k \to \infty}} k^{1/\rho} |a_k|^{1/k} = (\sigma e \rho)^{1/\rho} .$$

Among the entire functions of finite type one distinguishes entire functions of normal type $(\sigma > 0)$ and of minimal type $(\sigma = 0)$. If the condition (2) does not hold for any $\alpha < \infty$, then the function is said to be an entire function of maximal type or of infinite type. An entire function of order 1 and of finite type, and also an entire function of order less than 1, characterized by the condition

$$\overline{\lim_{k\to\infty}}\, k|a_k|^{1/k} = \beta < \infty,$$

is said to be of exponential type.

The zeros $z_1, z_2, \ldots,$ of an entire function $f(z)$ of order ρ have the property

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^{\rho+\epsilon}} < \infty, \qquad \text{for all } \epsilon > 0 .$$

Let p be the least integer $(p \leq \rho)$ such that $\sum_{k=1}^{\infty} |z_k|^{-p-1} < \infty$. Then the following product representation holds (see Hadamard theorem on entire functions)

$$f(z) = z^{\lambda} e^{P(z)} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) \exp\left(\frac{z}{z_k} + \ldots + \frac{z^p}{p z_k^p}\right),$$

where $P(z)$ is a polynomial of degree not exceeding ρ.

To characterize the growth of an entire function $f(z)$ of finite order ρ and finite type σ along rays, one introduces the quantity

$$h(\phi) = \overline{\lim_{r\to\infty}} \frac{\ln |f(re^{i\phi})|}{r^{\rho}}$$

— the growth indicator (cf. Growth indicatrix). Here, one always has

$$|f(re^{i\phi})| < e^{(h(\phi)+\epsilon)r^{\rho}}, \qquad r > r_0(\epsilon), \qquad \text{for all } \epsilon > 0 .$$

If

$$|f(re^{i\phi})| > e^{(h(\phi)-\epsilon)r^{\rho}}, \qquad r > r_0(\epsilon), \qquad z \notin E_0,$$

where E_0 is a set which is small in a certain sense (a set of relative measure 0), then the zeros of $f(z)$ are distributed in the plane very regularly in a certain sense, and there is a precise relation between $h(\phi)$ and the characteristic (the density) of the zeros. A function $f(z)$ with this property is said to be a function of completely regular growth.

A function of several variables $f(z_1, \ldots, z_n)$ is entire if it is analytic for $|z_k| < \infty$ $(k = 1, \ldots, n)$. Again one may introduce the concepts of order and type (conjugate orders and types). A simple representation in the form of an infinite product is not available here, because in contrast to the case $n=1$ the zeros of $f(z)$ are not isolated.

Check in Progress-I

Note : Please give solution of questions in space give below:

Q. 1 State Hadamard's Factorization Theorem.

Solution :

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Q. 2 Define Entire Function.

Solution :

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13.3 LAGUERRE'S METHOD

In numerical analysis, **Laguerre's method** is a root-finding algorithm tailored to polynomials. In other words, Laguerre's method can be used to numerically solve the equation $p(x) = 0$ for a given polynomial $p(x)$. One of the most useful properties of this method is that it is, from extensive empirical study, very close to being a "sure-fire" method, meaning that it is almost guaranteed to always converge to *some* root of the polynomial, no matter what initial guess is chosen. However, for computer computation, more efficient methods are known, with which it is guaranteed to find all roots (see Root-finding algorithm § Roots of polynomials) or all real roots (see Real-root isolation).

This method is named in honour of Edmond Laguerre, a French mathematician.

13.3.1 Laguerre's Polynomials

Polynomials that are orthogonal on the interval $(0, \infty)$ with weight function $\phi(x) = x^{\alpha} e^{-x}$, where $\alpha > -1$. The standardized Laguerre polynomials are defined by the formula

$$L_n^{\alpha}(x) = \frac{x^{-\alpha} e^{x}}{n!} \frac{d^n}{dx^n}(x^{\alpha+n} e^{-x}), \qquad n = 0, 1, \ldots .$$

Their representation by means of the gamma-function is

$$L_n^{\alpha}(x) = \sum_{k=0}^{n} \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+k+1)} \frac{(-x)^k}{k!(n-k)!} .$$

In applications the most important formulas are:

$$(n+1)L_{n+1}^{\alpha}(x) = (\alpha+2n+1-x)L_n^{\alpha}(x) - (\alpha+n)L_{n-1}^{\alpha}(x),$$
$$xL_{n-1}^{\alpha+1}(x) = (n+\alpha)L_{n-1}^{\alpha}(x) - nL_n^{\alpha}(x),$$
$$(L_n^{\alpha}(x))' = -L_{n-1}^{\alpha+1}(x) .$$

The polynomial $L_n^{\alpha}(x)$ satisfies the differential equation (Laguerre equation)

$$xy'' + (\alpha - x + 1)y' + ny = 0, \qquad n = 1, 2, \ldots .$$

The generating function of the Laguerre polynomials has the form

$$\frac{e^{-xt/(1-t)}}{(1-t)^{\alpha+1}} = \sum_{n=0}^{\infty} L_n^{\alpha}(x) t^n .$$

The orthonormal Laguerre polynomials can be expressed in terms of the standardized polynomials as follows:

$$\hat{L}_n^{\alpha}(x) = (-1)^n L_n^{\alpha}(x) \sqrt{\frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)}} .$$

The set of all Laguerre polynomials is dense in the space of functions whose square is integrable with weight $\phi(x)$ on the interval $(0, \infty)$.

Laguerre polynomials are most frequently used under the condition $\alpha = 0$; these were investigated by E. Laguerre [1], and are denoted in this case by $L_n(x)$ (in contrast to them, the $L_n^{\alpha}(x)$ are sometimes known as generalized Laguerre polynomials). The first few Laguerre polynomials $L_n(x)$ have the form

$$L_0(x) = 1, \qquad L_1(x) = 1 - x,$$
$$L_2(x) = 1 - 2x + \frac{x^2}{2},$$
$$L_3(x) = 1 - 3x + \frac{3x^2}{2} - \frac{x^3}{6},$$
$$L_4(x) = 1 - 4x + 3x^2 - \frac{2x^3}{3} + \frac{x^4}{24}.$$

The Laguerre polynomial $L_n^{\alpha}(x)$ is sometimes denoted by $L_n(x;\ \alpha)$.

The Laguerre polynomials are solutions $L_n(x)$ to the Laguerre differential equation with $\nu = 0$. They are illustrated above for $x \in [0, 1]$ and $n = 1$, 2, ..., 5, and implemented in the Wolfram Language as LaguerreL[*n*, *x*].

The first few Laguerre polynomials are

$$L_0(x) = 1 \tag{1}$$
$$L_1(x) = -x + 1 \tag{2}$$
$$L_2(x) = \frac{1}{2}\left(x^2 - 4x + 2\right) \tag{3}$$
$$L_3(x) = \frac{1}{6}\left(-x^3 + 9x^2 - 18x + 6\right). \tag{4}$$

When ordered from smallest to largest powers and with the denominators factored out, the triangle of nonzero coefficients is 1; -1, 1; 2, -4, 1; -6, 18, -9 1; 24, -96, ... (OEIS A021009). The leading denominators are 1, -1, 2, -6, 24, -120, 720, -5040, 40320, -362880, 3628800, ...

The Laguerre polynomials are given by the sum

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n}{k} x^k, \tag{5}$$

where $\binom{n}{k}$ is a binomial coefficient.

The Rodrigues representation for the Laguerre polynomials is

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}\left(x^n e^{-x}\right) \tag{6}$$

and the generating function for Laguerre polynomials is

$$g(x,z) = \frac{\exp\left(-\frac{xz}{1-z}\right)}{1-z} \tag{7}$$

$$= 1 + (-x+1)z + \left(\tfrac{1}{2}x^2 - 2x + 1\right)z^2 + \left(-\tfrac{1}{6}x^3 + \tfrac{3}{2}x^2 - 3x + 1\right)z^3 + \ldots \tag{8}$$

A contour integral that is commonly taken as the definition of the Laguerre polynomial is given by

$$L_n(z) = \frac{1}{2\pi i} \oint \frac{e^{-zt/(1-t)}}{(1-t)\,t^{n+1}}\,dt, \tag{9}$$

where the contour γ encloses the origin but not the point $z = 1$ (Arfken 1985, pp. 416 and 722).

The Laguerre polynomials satisfy the recurrence relations

$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x) \tag{10}$$

(Petkovšek *et al.* 1996) and

$$x L_n'(x) = nL_n(x) - nL_{n-1}(x). \tag{11}$$

Solutions to the associated Laguerre differential equation with $\nu \neq 0$ and k an integer are called associated Laguerre polynomials $L_n^k(x)$

13.3.2 Laguerre's Transform

The integral transform

$$f(n) = T\{F(x)\} = \int_0^\infty e^{-x} L_n(x) F(x)\,dx, \qquad n = 0, 1, \ldots,$$

where $L_n(x)$ is the Laguerre polynomial (cf. Laguerre polynomials) of degree n. The inversion formula is

$$T^{-1}\{f(n)\} = F(x) = \sum_{n=0}^{\infty} f(n) L_n(x), \qquad 0 < x < \infty,$$

if the series converges. If F is continuous, F' is piecewise continuous on $[0,\ \infty)$ and $F(x)=O(e^{ax})$, $x\to\infty$, $a<1$, then

$$T\left\{\frac{dF(x)}{dx}\right\}=\sum_{k=0}^{n} f(k)-F(0),\qquad n=0,1,...,$$

$$T\left\{x\frac{dF(x)}{dx}\right\}=-(n+1)f(n+1)+nf(n),\qquad n=0,1,\ldots.$$

If F and F' are continuous, F'' is piecewise continuous on $[0,\ \infty)$ and $|F(x)|+|F'(x)|=O(e^{ax})$, $x\to\infty$, $a<1$, then

$$T\left\{e^{x}\frac{d}{dx}\left[xe^{-x}\frac{dF(x)}{dx}\right]\right\}=-nf(n),\qquad n=0,1,\ .\ .\ .\ .$$

If F is piecewise continuous on $[0,\ \infty)$ and $F(x)=O(e^{ax})$
, $x\to\infty$, $a<1$, then for

$$G(x)=\int_0^x F(t)\,dt,$$

$$g(n)=T\left\{\int_0^x F(t)\,dt\right\}=f(n)-f(n-1),\qquad n=1,2,...,$$

and for $n=0$,

$$g(0)=f(0)\,.$$

Suppose that F and G are piecewise continuous on $[0,\ \infty)$ and that

$$|F(x)|+|G(x)|=O(e^{ax}),\qquad x\to\infty,\qquad a<\frac{1}{2},$$
$$T\{F\}=f(n),\qquad T\{G\}=g(n)\,.$$

Then

$$T^{-1}\{f(n)g(n)\}=$$
$$=\frac{1}{\pi}\int_0^{\infty}e^{-t}F(t)\int_0^{\pi}e^{\sqrt{xt}\cos\theta}\cos(\sqrt{xt}\sin\theta)\times$$
$$\times G(x+t-2\sqrt{xt}\cos\theta)\,d\theta\,dt\,.$$

The generalized Laguerre transform is

$$f_{\alpha}(n)=T_{\alpha}\{F(x)\}=$$

$$= \int_0^\infty e^{-x} x^\alpha L_n^\alpha(x) F(x)\, dx, \qquad n = 0, 1, \ldots,$$

where $L_n^\alpha(x)$ is the generalized Laguerre polynomial

13.3.3 Laguerre Functions

Functions that are solutions of the equation

$$xy'' + (\alpha - x + 1)y' + ny = 0,$$

where α and n are arbitrary parameters. Laguerre functions can be expressed in terms of the degenerate hypergeometric function or in terms of Whittaker functions. For $n = 0, 1, \ldots,$ the solutions of equation (*) are called Laguerre polynomials. The function

$$e_n^{(\alpha)}(x) = x^{\alpha/2} e^{-x/2} L_n^\alpha(x),$$

where $L_n^\alpha(x)$ is a Laguerre polynomial, is sometimes also called a Laguerre function.

13.3.4 Laguerre's Differential Equation

The Laguerre differential equation is given by

$$x y'' + (1 - x) y' + \lambda y = 0. \qquad (1)$$

Equation (1) is a special case of the more general associated Laguerre differential equation, defined by

$$x y'' + (\nu + 1 - x) y' + \lambda y = 0 \qquad (2)$$

where λ and ν are real numbers (Iyanaga and Kawada 1980, p. 1481; Zwillinger 1997, p. 124) with $\nu = 0$.

The general solution to the associated equation (2) is

$$t = C_1\, U(-\lambda, 1 + \nu, x) + C_2\, L_\lambda^\nu(x), \qquad (3)$$

where $U(a, b, x)$ is a confluent hypergeometric function of the first kind and $L_\lambda^\nu(x)$ is a generalized Laguerre polynomial.

Note that in the special case $\lambda = 0$, the associated Laguerre differential equation is of the form

$$y''(x) + P(x)\,y'(x) = 0, \tag{4}$$

so the solution can be found using an integrating factor

$$\mu = \exp\left(\int P(x)\,dx\right) \tag{5}$$
$$= \exp\left(\int \frac{\nu + 1 - x}{x}\,dx\right) \tag{6}$$
$$= \exp\left[(\nu + 1)\ln x - x\right] \tag{7}$$
$$= x^{\nu+1}\,e^{-x}, \tag{8}$$

as

$$y = C_1 \int \frac{dx}{\mu} + C_2 \tag{9}$$
$$= C_1 \int \frac{e^x}{x^{\nu+1}}\,dx + C_2 \tag{10}$$
$$= C_2 - C_1\,x^{-\nu}\,E_{1+\nu}(-x), \tag{11}$$

where $E_n(x)$ is the E*n*-function.

The associated Laguerre differential equation has a regular singular point at 0 and an irregular singularity at ∞. It can be solved using a series expansion,

$$x\sum_{n=2}^{\infty} n(n-1)\,a_n\,x^{n-2} + (\nu+1)\sum_{n=1}^{\infty} n\,a_n\,x^{n-1} - x\sum_{n=1}^{\infty} n\,a_n\,x^{n-1} + \lambda\sum_{n=0}^{\infty} a_n\,x^n = 0 \tag{12}$$
$$\sum_{n=2}^{\infty} n(n-1)\,a_n\,x^{n-1} + (\nu+1)\sum_{n=1}^{\infty} n\,a_n\,x^{n-1} - \sum_{n=1}^{\infty} n\,a_n\,x^n + \lambda\sum_{n=0}^{\infty} a_n\,x^n = 0 \tag{13}$$
$$\sum_{n=1}^{\infty} (n+1)\,n\,a_{n+1}\,x^n + (\nu+1)\sum_{n=0}^{\infty} (n+1)\,a_{n+1}\,x^n - \sum_{n=1}^{\infty} n\,a_n\,x^n + \lambda\sum_{n=0}^{\infty} a_n\,x^n = 0 \tag{14}$$
$$[(\nu+1)\,a_1 + \lambda\,a_0] + \sum_{n=1}^{\infty}\{[(n+1)\,n + (\nu+1)(n+1)]\,a_{n+1} - n\,a_n + \lambda\,a_n\}\,x^n = 0 \tag{15}$$
$$[(\nu+1)\,a_1 + \lambda\,a_0] + \sum_{n=1}^{\infty}[(n+1)(n+\nu+1)\,a_{n+1} + (\lambda - n)\,a_n]\,x^n = 0. \tag{16}$$

This requires

$$a_1 = -\frac{\lambda}{\nu + 1} a_0 \tag{17}$$

$$a_{n+1} = \frac{n - \lambda}{(n + 1)(n + \nu + 1)} a_n \tag{18}$$

for $n > 1$. Therefore,

$$a_{n+1} = \frac{n - \lambda}{(n + 1)(n + \nu + 1)} a_n \tag{19}$$

for $n = 1$, 2, ..., so

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{20}$$

$$= a_0 \, {}_1F_1 (-\lambda, \nu + 1, x) \tag{21}$$

$$= a_0 \left[1 - \frac{\lambda}{\nu + 1} x - \frac{\lambda (1 - \lambda)}{2 (\nu + 1)(\nu + 2)} x^2 - \frac{\lambda (1 - \lambda)(2 - \lambda)}{2 \cdot 3 (\nu + 1)(\nu + 2)(\nu + 3)} x^3 + \cdots \right] \tag{22}$$

If λ is a nonnegative integer, then the series terminates and the solution is given by

$$y = a_0 \frac{\lambda! \, L_\lambda^\nu (x)}{(\nu + 1)_\lambda}, \tag{23}$$

where $L_\lambda^\nu (x)$ is an associated Laguerre polynomial and $(a)_n$ is a Pochhammer symbol. In the special case $\nu = 0$, the associated Laguerre polynomial collapses to a usual Laguerre polynomial and the solution collapses to

$$y = a_0 \, L_\lambda (x).$$

Check in Progress-II

Note : Please give solution of questions in space give below:

Q. 1 Define Laguerre's polynomials .

Solution :

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Q. 2 Define Laguerre's Differential Equation .

Solution :

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13.4 CONTOUR INTEGRAL

An integral obtained by contour integration. The particular path in the complex plane used to compute the integral is called a contour.

As a result of a truly amazing property of holomorphic functions, a closed contour integral can be computed simply by summing the values of the complex residues *inside* the contour.

Watson uses the notation $\int^{(a+)} f(z)\,dz$ to denote the contour integral of $f(z)$ with contour encircling the point a once in a counterclockwise direction.

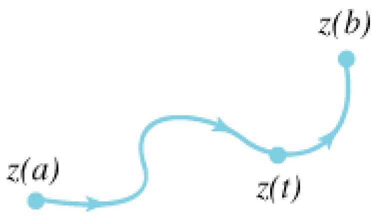
we learned how to evaluate integrals of the form $\int_a^b f(t)\,dt$, where f(t) was complex-valued and $[a, b]$ was an interval on the real axis (so that t was real, with $t \in [a, b]$). In this section, we define and evaluate integrals of the form $\int_C f(z)\,dz$, where f(t) is complex-valued and C is a contour in the plane (so that z is complex, with $z \in C$).

Recall that to represent a curve C we used the parametric notation

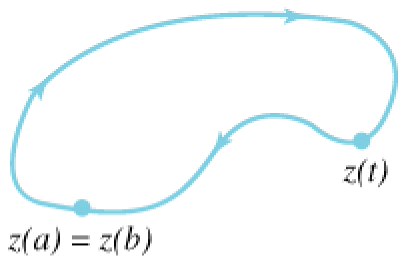
(1) $C: z(t) = x(t) + i\,y(t)$ for $a \le t \le b$,

where x(t) and y(t) are continuous functions. We now place a few more restrictions on the type of curve to be described. The following discussion leads to the concept of a contour, which is a type of curve that is adequate for the study of integration.

Recall that C is simple if it does not cross itself, which means that $z(t_1) \neq z(t_2)$ whenever $t_1 \neq t_2$, except possibly when $t_1 = a$ and $t_2 = b$. A curve C with the property that $z(a) = z(b)$ is a closed curve.
If $z(a) = z(b)$ is the only point of intersection, then we say that C is a simple closed curve. As the parameter t increases from the value $t = a$ to the value $t = b$, the point $z(t)$ starts at the initial point $z(a)$, moves along the curve C , and ends up at the terminal point $z(b)$. If C is simple, then $z(t)$ moves continuously from $z(a)$ to $z(b)$ as t increases and the curve is given an orientation, which we indicate by drawing arrows along the curve. Figure 6.1 illustrates how the terms simple and closed describe a curve.

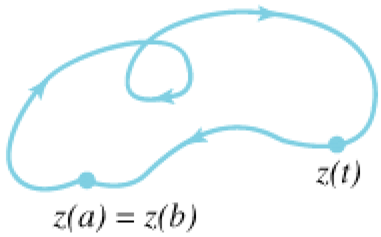


(a) A curve that is simple.



(b) A simple closed curve.

(c) A curve that is *not* simple and *not* closed.



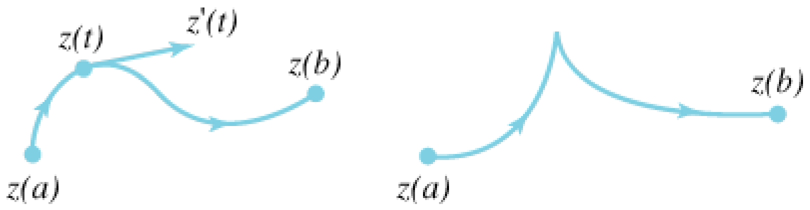
(d) A closed curve that is *not* simple.

Figure 13.1 The terms simple and closed used to describe curves.

The complex-valued function $z(t) = x(t) + iy(t)$ is said to be differentiable on $[a, b]$ if both $x(t)$ and $y(t)$ are differentiable for $a \le t \le b$. Here we require the one-sided derivatives of $x(t)$ and $y(t)$ to exist at the endpoints of the interval. As the derivative $z'(t)$ is

$z'(t) = x'(t) + iy'(t)$ for $a \le t \le b$.

The curve C defined is said to be a smooth curve if $z'(t)$ is continuous and nonzero on the interval. If C is a smooth curve, then C has a nonzero tangent vector at each point $z(t)$, which is given by the vector $z'(t)$. If $x'(t_0) = 0$, then the tangent vector $z'(t_0) = iy'(t_0)$ is vertical. If $x'(t_0) \ne 0$, then the slope $\frac{dy}{dx}$ of the tangent line to C at the point $z(t_0)$ is given by $\frac{y'(t_0)}{x'(t_0)}$. Hence for a smooth curve the angle of inclination $\theta(t)$ of its tangent vector $z'(t)$ is defined for all values of $t \in [a, b]$ and is continuous. Thus a smooth curve has no corners or cusps. Figure 6.2 illustrates this concept.



(a) A smooth curve. (b) A curve that is *not* smooth.

Figure 13.2 The term smooth used to describe curves.

If C is a smooth curve, then ds, the differential of arc length, is given by

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = |z'(t)|\, dt.$$

The function $s(t) = \sqrt{[x'(t)]^2 + [y'(t)]^2}$ is continuous, as $x'(t)$ and $y'(t)$ are continuous functions, so the <u>length</u> $L(C)$ of the curve C is

(2) $$L(C) = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = \int_a^b |z'(t)|\, dt.$$

Now consider C to be a curve with parameterization

$C: z_1(t) = x(t) + i y(t)$ for $a \le t \le b$.

The opposite curve $-C$ traces out the same set of points in the plane, but in the reverse order, and has the parametrization

$-C: z_2(t) = x(-t) + i y(-t)$ for $a \le t \le b$.

Since $z_2(t) = z_1(-t)$, $-C$ is merely C traversed in the opposite sense, as illustrated in Figure 6.3.

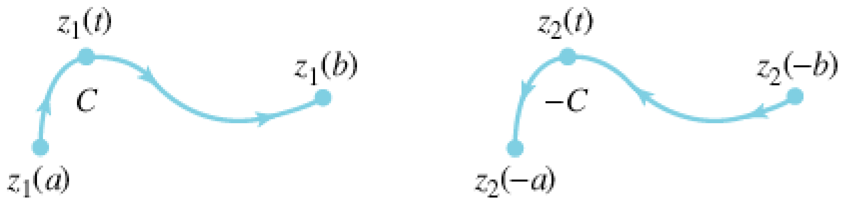


Figure 13.3 The curve C and its opposite curve $-C$.

A curve C that is constructed by joining finitely many smooth curves end to end is called a <u>contour</u>. Let $C_1, C_2, \ldots, C_n$ denote n smooth curves such that the terminal point of the curve C_k coincides with the initial point of C_{k+1} for $k = 1, 2, \ldots, n-1$. We express the contour C by the equation

$$C = C_1 + C_2 + \ldots + C_n.$$

A synonym for contour is path.

Example . Find a parameterization of the polygonal path $C = C_1 + C_2 + C_3$ from $-1 + i$ to $3 - i$ shown in Figure 6.4.

Here C_1 is the line from $-1 + i$ to -1, C_2 is the line from -1 to $1 + i$, and C_3 is the line from $1 + i$ to $3 - i$.

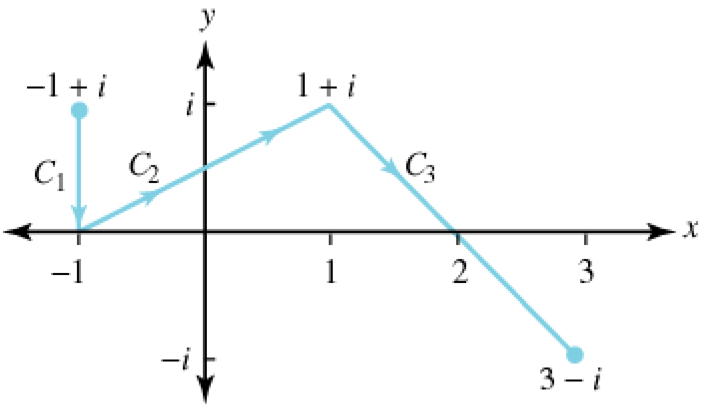


Figure 13.4 The polygonal path $C = C_1 + C_2 + C_3$ from $-1 + i$ to $3 - i$.

Solution. We express C as three smooth curves, or $C = C_1 + C_2 + C_3$. If we set $z_0 = -1 + i$ and $z_1 = -1$, we can use to get a formula for the straight-line segment joining two points:

$$C_1 : \; z_1(t) \; = \; z_0 + t(z_1 - z_0) \; = \; (-1 + i) + t(-1 - (-1 + i)),$$

for $0 \le t \le 1$. When simplified, this formula becomes

$$C_1 : \; z_1(t) \; = \; -1 + i(1 - t), \text{ for } 0 \le t \le 1.$$

Similarly, the segments C_2 and C_3 are given by

$$C_2 : \; z_2(t) \; = \; -1 + 2t + it, \text{ for } 0 \le t \le 1, \text{ and}$$

$$C_3 : \; z_3(t) \; = \; 1 + 2t + i(1 - 2t), \text{ for } 0 \le t \le 1.$$

We are now ready to define the integral of a complex function along a contour C in the plane with initial point A and terminal point B. Our approach is to mimic what is done in calculus. We create a partition $P_n = \{z_0 = A, z_1, z_2, \ldots, z_n = B\}$ of points that proceed along C from A to B and form the differences $\Delta z_k = z_k - z_{k-1}$ for $k = 1, 2, \ldots, n$. Between each pair of partition points z_{k-1} and z_k we select a point c_k on C, as shown in Figure 6.5, and evaluate the function

$f(c_k)$.

These values are used to make a Riemann Sum for the partition:

$$S(P_n) = \sum_{k=1}^{n} f(c_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} f(c_k)\Delta z_k.$$

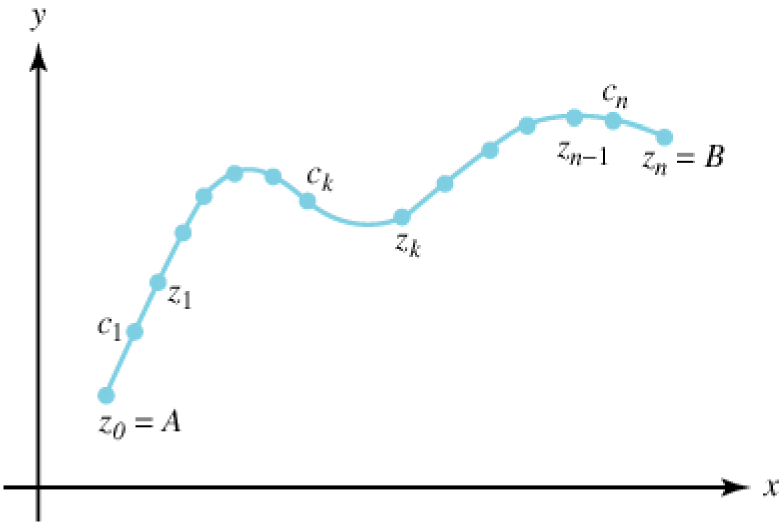


Figure 13.5 Partition points $\{z_k\}$ and function evaluation points $\{c_k\}$ for a Riemann sum along the contour C from $z = A$ to $z = B$.

Assume now that there exists a unique complex number L that is the limit of every sequence $\{S(P_n)\}$ of Riemann sums given , where the maximum of $|\Delta z_k|$ tends toward 0 for the sequence of partitions. We define the number L as the value of the integral of the function f(z) taken along the contour C.

13.4.1 Complex Integral

Definition (Complex Integral). Let C be a contour. Then

$$\int_C f(z)\,dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta z_k,$$

provided that the limit exists in the sense previously discussed.

Note the value of the integral depends on the contour. In the Cauchy-Goursat theorem will establish the remarkable fact that, if f(z) is analytic, then $\int_C f(z)\,dz$ is independent of the contour.

Example . Give an exact calculation of the integral in Example 6.6: $\int_C e^z\,dz$ where C is a the line segment joining the point $A = 0 \text{ to } B = 2 + \frac{i\pi}{4}$.

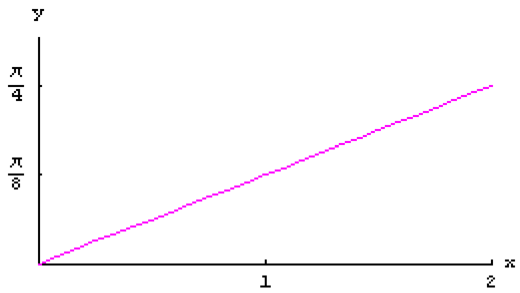


Figure 13.6

Solution. We must compute $\int_C e^z\,dz = \int_C \exp(z)\,dz$, where C is the line segment joining $A = 0 \text{ to } B = 2 + \frac{i\pi}{4}$, we can parametrize C by $z(t) = \left(2 + i\frac{\pi}{4}\right)t$, for $0 \le t \le 1$. As $z'(t) = \left(2 + i\frac{\pi}{4}\right)$, Theorem 6.1 guarantees that

$$\int_C e^z\,dz = \int_C \exp(z)\,dz = \int_0^1 \text{Exp}[z[t]]\,z'[t]\,dt$$

$$= \int_0^1 \text{Exp}\left[\left(2 + i\frac{\pi}{4}\right)t\right]\left(2 + i\frac{\pi}{4}\right)dt$$

$$= \left(2 + i\frac{\pi}{4}\right)\int_0^1 \text{Exp}\left[\left(2 + i\frac{\pi}{4}\right)t\right]dt$$

$$= \left(2 + i\frac{\pi}{4}\right)\int_0^1 e^{2t} e^{\frac{i\pi}{4}t}\,dt$$

$$= \left(2 + i\frac{\pi}{4}\right)\int_0^1 e^{2t}\left(\text{Cos}\left[\frac{\pi t}{4}\right] + i\,\text{Sin}\left[\frac{\pi t}{4}\right]\right)dt$$

$$= \left(2 + i\frac{\pi}{4}\right)\left(\int_0^1 e^{2t}\,\text{Cos}\left[\frac{\pi t}{4}\right]dt + i\int_0^1 e^{2t}\,\text{Sin}\left[\frac{\pi t}{4}\right]dt\right)$$

$$= \left(2 + i\frac{\pi}{4}\right)\left(\frac{2\sqrt{2}\,e^2\,(\pi+8)}{\pi^2+64} - \frac{32}{\pi^2+64} + i\left(\frac{-2\sqrt{2}\,e^2\,(\pi-8)}{\pi^2+64} + \frac{4\pi}{\pi^2+64}\right)\right)$$

$$= 4.2248516741 + 5.2248516741\,i$$

Example . Evaluate the contour integral $\int_C \frac{1}{z-2}\,dz$ where C is a the upper semicircle with radius 1 centered at $x = 2$.

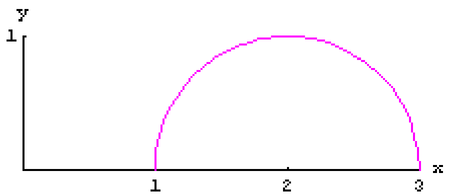


Figure 13.7

Solution. The function $z(t) = 2 + e^{it}$, for $0 \le t \le \pi$ is a parametrization for C. We apply Theorem 6.1 with $f(z) = \frac{1}{z-2}$.

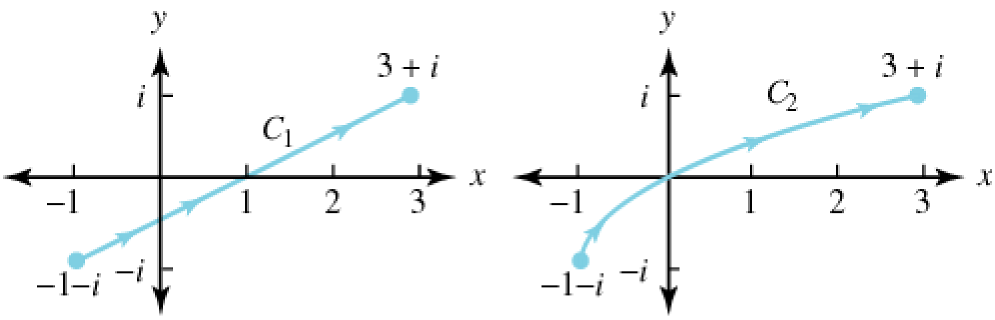
(Note: $f(z(t)) = \frac{1}{z(t) - 2}$), and $z'(t) = i e^{it}$.) Hence

$$\int_C \frac{1}{z-2}\,dz = \int_0^\pi \frac{1}{z(t)-2}\,z'(t)\,dt = \int_0^\pi \frac{1}{2+e^{it}-2}\,i e^{it}\,dt = \int_0^\pi i\,dt = i\pi$$

.

Example . Show that $\int_{C_1} z\,dz = \int_{C_2} z\,dz = 4 + 2i$,

where C_1 is the line segment from $-1 - i$ to $3 + i$, and C_2 is the portion of the parabola $x = y^2 + 2y$ joining $-1 - i$ to $3 + i$, as indicated in Figure 6.8.



(a) The line segment.

(b) The portion of the parabola.

Figure 13.8 The two contours C_1 and C_2 joining $-1 - i$ to $3 + i$.

Solution. The line segment joining $(-1, -1)$ to $(3, 1)$ is given by the slope intercept formula $y = \frac{1}{2}x - \frac{1}{2}$, which can be written as $x = 2y + 1$.

If we choose the parametrization $y = t$ and $x = 2t + 1$, we can write segment C_1 as

$C_1 : z(t) = 2t + 1 + it$ and $dz = (2 + i)\,dt$

for $-1 \le t \le 1$.

Along C_1 we have $f(z(t)) = 2t + 1 + it$. Applying Theorem 6.1 gives

$$\int_{C_1} z\,dz = \int_a^b \frac{1}{z(t) - 2} z'(t)\,dt = = \int_{-1}^{1} (2t + 1 + it)(2 + i)\,dt .$$

We now multiply out the integrand and put it into its real and imaginary parts:

$$\int_{C_1} z\,dz = \int_{-1}^{1} (2t + 1 + it)(2 + i)\,dt$$

$$= \int_{-1}^{1} (3t + 2 + i(4t + 1))\,dt$$

$$= \int_{-1}^{1} (3t + 2)\,dt + i\int_{-1}^{1} (4t + 1)\,dt$$

$$= 4 + 2i$$

Similarly, we can parametrize the portion of the parabola $x = y^2 + 2y$ joining $(-1, -1)$ to $(3, 1)$ by $y = t$ and $x = 2t + 1$ and $x = t^2 + 2t$ so that

$C_2: \; z(t) = t^2 + 2t + it$ and $dz = (2t + 2 + i)\,dt$

for $-1 \le t \le 1$.

Along C_2 we have $f(z(t)) = t^2 + 2t + it$. Theorem 6.1 now gives

$$\int_{C_2} z\,dz = \int_{-1}^{1} (t^2 + 2t + it)(2t + 2 + i)\,dt$$

$$= \int_{-1}^{1} (2t^3 + 6t^2 + 3t + i(3t^2 + 4t))\,dt$$

$$= \int_{-1}^{1} (2t^3 + 6t^2 + 3t)\,dt + i\int_{-1}^{1} (3t^2 + 4t)\,dt$$

$$= 4 + 2i$$

13.5 SUMMARY

We study in this unit Genus theorem with statement. We study Hadamard's Factorization Theorem with its statement and its proof. We study Cauchy-Hadamard Theorem and its proof with statement. We study Genus of an Entire Function.We study Entire function with its proof. We study Laguerre's method , Laguerre's Polynomials and its properties. We study Laguerre's Differential Equation.

13.6 KEYWORD

Genus : a principal taxonomic category that ranks above species and below family, and is denoted by a capitalized Latin name

Entire : with no part left out; whole.

Laguerre's : The algorithm of the *Laguerre* method to find one root of a ... larger absolute value, to avoid loss of *significance* as iteration proceeds

13.7 QUESTIONS FOR REVIEW

Exercise 1. Consider the integral $\int_C z^2\, dz$, where C is the positively oriented upper semicircle of radius 1, centered at 0.

Give a Riemann sum approximation for the integral by selecting $n = 4$ and using the

points

$z_k = e^{i\frac{k\pi}{4}}$ (for $k = 0, 1, 2, 3, 4$) and $c_k = e^{i\frac{(2k-1)\pi}{4}}$ (for $k = 1, 2, 3, 4$)

Exercise 2. Show that the integral $\int_C e^z\, dz$ where C is a the line segment joining the point $A = 0$ to $B = 2 + \frac{i\pi}{4}$

simplifies to $\int_C e^z\, dz = \exp\left(2 + i\frac{\pi}{4}\right) - 1$

Exercise 3. Recall $C_r^+(a)$ is the circle of radius r centered at a, oriented counter-clockwise.

Evaluate $\int_{C_4^+(0)} z\, dz$.

Exercise 4. Evaluate $\int_{C_2^-(0)} \frac{1}{z}\, dz$. (The minus sign in $C_2^-(0)$ means the clockwise orientation.)

Ecercise 5 Evaluate $\int_C (z + 1)\, dz$, where C is the portion of $C_1^+(0)$ in the first quadrant.

Exercise 6 Evaluate $\int_C (x^2 - i y^2)\, dz$, where C s the upper half of $C_1^+(0)$.

Exercise 7. Let $f(z)$ be a continuous function on the circle $C_R^+(z_0) = \{z : |z - z_0| = R\}$.

Show that $\int_{C_R^+(z_0)} f(z)\, dz = i R \int_0^{2\pi} f\left(z_0 + R e^{i\theta}\right) e^{i\theta}\, d\theta$.

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13.9 ANSWER TO CHECK YOUR PROGRESS

Check In Progress-I

Answer Q. 1 Check in Section 1.2

2 Check in Section 2.1

Check In Progress-II

Answer Q. 1 Check in section 3.1

2 Check in Section 3.4

UNIT 14: BOREL'S AND PICARD'S THEOREM

STRUCTURE

14.0 Objective

14.1 Introduction

14.1.1 Borel Fixed-Point Theorem

14.1.2 Borel Function

14.1.3 Fourier-Borel Transform

14.2 Picard's Theorem

14.2.1 Casorati-Sokhotskii-Weierstrass Theorem

14.2.2 Meromorphic Function

14.2.3 Casorati-Sokhotskii-Weierstrass Theorem

14.2.4 Weierstrass Theorem

14.2.5 Riemann-Roch Theorem

14.3 Cauchy-Riemann Equation

14.4 Summary

14.5 Keyword

14.6 Questions for review

14.7 Suggestion Reading and References

14.8 Answer to check your Progress

14.0 OBJECTIVE

- Deals with Borel fixed-Point Theorem
- Deals with Borel function and its statement
- Deals with Fourier-Borel transform
- Deals with Casorati-Sokhotskii-Weierstrass Theorem
- Deals with Meromorphic Function
- Deals with Weierstrass Theorem and its statement with proof
- Deals with Riemann-Roch Theorem
- Deals with Cauchy-Riemann Theorem with examples and proof

14.1 INTRODUCTION

In mathematics, the Borel–Carathéodory theorem in complex analysis shows that an analytic function may be bounded by its real part. It is an application of the maximum modulus principle. It is named for Émile Borel and Constantin Carathéodory.

In complex analysis, Picard's great theorem and Picard's little theorem are related theorems about the range of an analytic function. They are named after Émile Picard.

Little Picard Theorem: If a function $f : C \to C$ is entire and non-constant, then the set of values that $f(z)$ assumes is either the whole complex plane or the plane minus a single point.

Sketch of Proof: Picard's original proof was based on properties of the modular lambda function, usually denoted by λ, and which performs, using modern terminology, the holomorphic universal covering of the twice punctured plane by the unit disc. This function is explicitly constructed in the theory of elliptic functions. If *f* omits two values, then the composition of *f* with the inverse of the modular function maps the plane into the unit disc which implies that *f* is constant by Liouville's theorem.

This theorem is a significant strengthening of Liouville's theorem which states that the image of an entire non-constant function must be unbounded. Many different proofs of Picard's theorem were later found and Schottky's theorem is a quantitative version of it. In the case where the values of *f* are missing a single point, this point is called a lacunary value of the function.

Great Picard's Theorem: If an analytic function *f* has an essential singularity at a point *w*, then on any punctured neighborhood of *w*, $f(z)$ takes on all possible complex values, with at most a single exception, infinitely often.

This is a substantial strengthening of the Casorati–Weierstrass theorem, which only guarantees that the range of *f* is dense in the complex plane.

A result of the Great Picard Theorem is that any entire, non-polynomial function attains all possible complex values infinitely often, with at most one exception.

The "single exception" is needed in both theorems, as demonstrated here:

- e^z is an entire non-constant function that is never 0,
- $e^{1/z}$ has an essential singularity at 0, but still never attains 0 as a value.

14.1.1 Borel Fixed-Point Theorem

A connected solvable algebraic group G acting regularly (cf. Algebraic group of transformations) on a non-empty complete algebraic variety V over an algebraically-closed field k has a fixed point in V. It follows from this theorem that Borel subgroups of algebraic groups are conjugate (The Borel–Morozov theorem). The theorem was demonstrated by A. Borel [1]. Borel's theorem can be generalized to an arbitrary (not necessarily algebraically-closed) field k: Let V be a complete variety defined over a field k on which a connected solvable k-split group G acts regularly, then the set of rational k-points V(k) is either empty or it contains a point which is fixed with respect to G. Hence the generalization of the theorem of conjugation of Borel subgroup is: If the field k is perfect, the maximal connected solvable k-split subgroups of a connected k-defined algebraic group H are mutually conjugate by elements of the group of k-points of H.

14.1.2 Borel Function

Definition

A map f:X→Y between two topological spaces is called Borel (or Borel measurable) if f−1(A) is a Borel set for any open set A (recall that the σσ-algebra of Borel sets of X is the smallest σσ-algebra containing the open sets). When the target Y is the real line, it suffices to assume that $f^{-1}(]a,\infty[)$ is Borel for any $a\in R$. Consider two topological spaces X and Y and the corresponding Borel σσ-algebras B(X) and B(Y). The Borel measurability of the function f:X→Y is then equivalent to the measurability of the map ff seen as map between the measurable spaces (X,B(X)) and (Y,B(Y)), see also Measurable mapping.

Properties

As it is always the case for measurable real functions on any measurable space X, the space of Borel real-valued functions over a given topological space is a vector space and it is closed under the operation of taking pointwise limits of sequences (i.e. if a sequence of Borel functions fn converges everywhere to a function f, then f is also a Borel function),

Closure under composition

Moreover the compositions of Borel functions of one real variable are Borel functions. Indeed, if X,Y and Z are topological spaces and f:X→Y, g:Y→Z Borel functions, then g∘f is a Borel function, as it follows trivially from the definition above.

Comparison with Lebesgue measurable functions

The latter property is false for real-valued Lebesgue measurable functions on R (cf. Measurable function): there are pairs of Lebesgue measurable functions f,g:R→R such that f∘g is not Lebesgue measurable (the Lebesgue measurability of f∘g holds if we assume in addition that f is continuous, whereas it fails if we assume the continuity of g but only the Lebesgue measurability of f,

All Borel real valued functions on the euclidean space are Lebesgue-measurable, but the converse is false. However, it follows easily from Lusin's Theorem that for any Lebesgue-measurable function ff there exists a Borel function gg which coincides with ff almost everywhere (with respect to the Lebesgue measure).

Comparison with Baire functions

Borel functions f:R→R are sometimes called Baire functions, since in this case the set of all Borel functions is identical with the set of functions belonging to the Baire classes (Lebesgue's theorem, [Hau]).

However, in the context of a general topological space X the space of Baire functions is the smallest family of real-valued functions which is close under the operation of taking limits of pointwise converging sequences and which contains the continuous functions. In a general topological space the class of Baire functions might be strictly smaller then the class of Borel functions.

Borel real-valued functions of one real variable can be classified by the order of the Borel sets; the classes thus obtained are identical with the Baire classes.

14.1.3 Fourier-Borel Transform

Let $\mathbf{C}^n$ be the n-dimensional complex space, and let $\mathcal{H}(\mathbf{C}^n)$ denote the space of entire functions in n complex variables, equipped with the topology of uniform convergence on the compact subsets of $\mathbf{C}^n$ (cf. also Entire function; Uniform convergence). Let $\mathcal{H}(\mathbf{C}^n)'$ be its dual space of continuous linear functionals. The elements of $\mathcal{H}(\mathbf{C}^n)'$ are usually called analytic functionals in $\mathbf{C}^n$.

One says that a compact set $K \subseteq \mathbf{C}^n$ is a carrier for an analytic functional $\mu \in \mathcal{H}(\mathbf{C}^n)'$ if for every open neighbourhood U of K there exists a positive constant C_U such that, for every $f \in \mathcal{H}(\mathbf{C}^n)$,

$$|\mu(f)| \leq C_U \sup_U |f(z)|.$$

General references for these notions are [a3], [a5].

Let $\mu \in \mathcal{H}(\mathbf{C}^n)'$. The Fourier–Borel transform $\mathcal{F}\mu(\zeta)$ is defined by

$$\mathcal{F}\mu(\zeta) = \mu(\exp \zeta z),$$

where $\zeta z = \zeta_1 z_1 + \ldots + \zeta_n z_n$

For $n = 1$, the use of this transform goes back to E. Borel, while for $n > 1$ it first appeared in a series of papers by A. Martineau, culminating with [a6].

It is immediate to show that $\mathcal{F}\mu$ is an entire function. Moreover, since the exponentials are dense in $\mathcal{H}(\mathbf{C}^n)$, an analytic functional is uniquely determined by its Fourier–Borel transform.

By using the definition of carrier of an analytic functional, it is easy to see that if $\mu \in \mathcal{H}(\mathbf{C}^n)'$ is carried by a compact convex set K, then for every $\epsilon > 0$ there exists a number $C_\epsilon > 0$ such that, for any $\zeta \in \mathbf{C}^n$,

$$|\mathcal{F}\mu(\zeta)| \leq C_\epsilon \exp(H_K(\zeta) + \epsilon|\zeta|),$$

where $H_K(\zeta) = \sup_{z\in K} \operatorname{Re}(\zeta z)$ is the support function of K.

A fundamental result in the theory of the Fourier–Borel transform is the fact that the converse is true as well: Let $f(\zeta)$ be an entire function. Suppose that for some compact convex set K and for every $\epsilon > 0$ there exists a number $C_\epsilon > 0$ such that, for any $\zeta \in \mathbf{C}^n$,

$$|f(\zeta)| \leq C_\epsilon \exp(H_K(\zeta) + \epsilon|\zeta|).$$

Then f is the Fourier–Borel transform of an analytic functional μ carried by K.

This theorem, for $n = 1$, was proved by G. Pólya, while for $n > 1$ it is due to A. Martineau [a7].

In particular, the Fourier–Borel transform establishes an isomorphism between the space $\mathcal{H}(\mathbf{C}^n)'$ and the space $\operatorname{Exp}(\mathbf{C}^n)$ of entire functions of exponential type, i.e. those entire functions f for which there are positive constants A, B such that

$$|f(\zeta)| \leq A \exp(B|\zeta|).$$

If $\mathcal{H}(\mathbf{C}^n)'$ is endowed with the strong topology, and $\operatorname{Exp}(\mathbf{C}^n)$ with its natural inductive limit topology, then the Fourier–Borel transform is actually a topological isomorphism, [a2].

A case of particular interest occurs when, in the above assertion, one takes $K = \{0\}$. In this case, a function which satisfies the estimate (a1), i.e.

$$|f(\zeta)| \leq C_\epsilon \exp(\epsilon|\zeta|)$$

is said to be of exponential type zero, or of infra-exponential type. Given such a function f, there exists a unique analytic functional μ such that $\mathcal{F}\mu = f$; such a functional is carried by $K = \{0\}$ and therefore is a continuous linear functional on any space $\mathcal{H}(U)$, for U an open subset of $\mathbf{C}^n$ containing the origin. If one denotes by $\mathcal{O}\{0\}$ the space of germs of holomorphic functions at the origin (cf. also Germ), then $\mathcal{O}'_{\{0\}} = \mathcal{B}_{\{0\}}$, the space of hyperfunctions supported at the origin (cf. also Hyperfunction); the Fourier–Borel transform is therefore well defined on such a space. In fact, it is well defined on every hyperfunction with compact support. For this and related topics, see e.g. [a1], [a4].

The Fourier–Borel transform is a central tool in the study of convolution equations in convex sets in $\mathbf{C}^n$. As an example, consider the problem of surjectivity. Let Ω be an open convex subset of $\mathbf{C}^n$ and let $\mu \in \mathcal{H}(\mathbf{C}^n)'$ be carried by a compact set K. Then the convolution operator

$$\mu * : \mathcal{H}(\Omega + K) \rightarrow \mathcal{H}(\Omega)$$

is defined by

$$\mu * f(z) = \mu(\zeta \mapsto f(z + \zeta)).$$

One can show (see [a5] or [a1] and the references therein) that if $\mathcal{F}\mu$ is of completely regular growth and the radial regularized indicatrix of $\mathcal{F}\mu$ coincides with H_K, then $\mu *$ is a surjective operator. The converse is true provided that Ω is bounded, strictly convex, with C^2 boundary.

Check in Progress-I

Note : Please give solution of questions in space give below:

Q. 1 Give definition of Borel Function.

Solution :

…………………………………………………………………………………

…………………………………………………………………………………

…..………………………………………………………………………………

Q. 2 State Fourier-Borel Transform .

Solution :

…………………………………………………………………………………

…………………………………………………………………………………

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14.2 PICARD'S THEOREM

Picard's theorem on the behaviour of an analytic function $f(z)$ of a complex variable z near an essential singular point α is a result in classical function theory that is the starting point of numerous profound researches. It consists of two parts: a) Picard's little theorem: Any entire function $f(z) \neq \mathrm{const}$ assumes any finite complex value with the possible exception of one value; and b) Picard's big theorem: Any single-valued analytic function $f(z)$ assumes any finite complex value, with the possible exception of one value, in an arbitrary neighbourhood around an isolated essential singular point α.

This theorem was first published by E. Picard ,

and it substantially supplements the Sokhotskii theorem. Picard's little theorem is a consequence of the big one. It follows directly from Picard's big theorem that any finite complex value, with the possible exception of one value, is assumed in an arbitrary neighbourhood of an essential singular point infinitely often. For a meromorphic function in the finite plane $\mathbf{C} = \{z : |z| < \infty\}$, Picard's theorem takes the form: If the point $\alpha = \infty$ is essentially singular for a function $F(z)$ that is meromorphic in $\mathbf{C}$, then in an arbitrary neighbourhood of α the function $F(z)$ assumes any complex value in the extended complex plane $\overline{\mathbf{C}} = \{z : |z| \leq \infty\}$, with the possible exception of two values, and moreover infinitely often. The examples of the entire function $e^{z} \neq 0$ and the meromorphic function $\tan z \neq i, -i$ show that all these assertions are precise. The exceptional values appearing in Picard's theorem are called Picard exceptional values.

Picard's theorem is substantially supplemented by the Iversen theorem and the Julia theorem, which show, respectively, that the Picard exceptional values are asymptotic values (cf. Asymptotic value) and that there exist Julia rays L starting at the essential singular point α and such that the non-exceptional values are taken infinitely often even in an arbitrary small sector having its vertex at α and L as symmetry axis.

The following two directions are characteristic in modern studies related to Picard's theorem. Let E be the set of essential singular points of a meromorphic function $F(z)$, i.e. $F(z)$ is a meromorphic function in a certain neighbourhood of any point $z_0 \notin E$, and suppose that the cluster set $C(z_0; F)$ of $F(z)$ at a point $z_0 \in E$ does not reduce to one value. Let $R(\alpha; F)$, $\alpha \in E$, be the set of those values $w \in \overline{\mathbf{C}}$ that are assumed infinitely often in any neighbourhood of α. Then Picard's theorem asserts that if α is an isolated point in E, the complement

$$CR(\alpha; F) = \overline{\mathbf{C}} \setminus R(\alpha; F)$$

has the Picard property, i.e. it consists of at most two points. V.V. Golubev established in 1916 that if the capacity of E is zero, $\mathbf{cap}\ E = 0$, then $CR(\alpha; F)$ has capacity zero for all $\alpha \in E$. It has not been completely determined (up till 1983) what minimal conditions must be imposed on E in order that the set $CR(\alpha; F)$ has the Picard property for all $\alpha \in E$. Examples show that on the one hand the condition $\mathbf{cap}\ E = 0$ is not sufficient, while on the other that there is a set E, $\mathbf{cap}\ E > 0$, outside which there do not exist meromorphic transcendental functions omitting four values , , .

The second direction is related to generalizations of Picard's theorem to analytic functions $f(z)$ of several complex variables $z = (z_1, \dots, z_n)$, $n \geq 1$. For $n = 1$, Picard's theorem can also be formulated as follows: Any holomorphic mapping $f: \mathbf{C} \to \mathbf{C}$ that omits at least two points is constant. However, in 1922, P. Fatou constructed an example of a non-singular holomorphic mapping (and even of a biholomorphic mapping) $f: \mathbf{C}^2 \to \mathbf{C}^2$ for which the set of exceptional values $\mathbf{C}^2 \setminus f(\mathbf{C}^2)$ contains a non-empty open set. This means that Picard's theorem (and even Sokhotskii's theorem) cannot be generalized directly to the case $n > 1$. Generalizations of Picard's theorem are possible if one starts, for example, from another formulation, which is somewhat artificial for $n = 1$: Any holomorphic mapping $F: \mathbf{C} \to \mathbf{C}P$ into the complex projective plane $\mathbf{C}P = \overline{\mathbf{C}}$ that omits at least three hyperplanes (i.e. points for $n = 1$) is constant. In particular, Green's theorem applies: Any holomorphic mapping $F: \mathbf{C}^m \to \mathbf{C}P^n$ that omits at least $2n + 1$ hyperplanes in general position is constant (cf. , ,).

Picard's theorem on the uniformization of algebraic curves: If an algebraic curve $\Phi(z, w) = 0$ has genus $g > 1$, then there exists no pair of meromorphic functions $z = f(t)$, $w = h(t)$ such that $\Phi(f(t), h(t)) \equiv 0$. In other words, uniformization of algebraic curves of genus $g > 1$ by means of meromorphic functions is impossible. On the other hand, one can always perform uniformization in the case $g = 1$ by means of (meromorphic) elliptic functions.

14.2.1 Casorati-Sokhotskii-Weierstrass Theorem

A theorem which characterizes isolated essential singularities of holomorphic functions of one complex variable

Theorem Let f:U→C be an holomorphic function and z0 a point for which U is a punctured neighborhood. Then either the limit

limz→z0f(x)

exists in the extended complex plane C¯, or otherwise the cluster set C(z0,f) (namely the set of points w∈C¯ for which there is a sequence zn→z0 with f(zn)→w is the entire C¯.
In the latter case, the singularity is called essential. When the limit exists, then z0z0 is either a removable singularity, in which case the limit belongs to C, or a pole. Removable singularities, poles and essential singularities can also be characterized using the Laurent series. The

assumption that ff is defined on a punctured neighborhood of z0 can be weakend (see Essential singular point). Instead there is no direct generalization to the case of holomorphic functions of several complex variables.

The Casorati-Sokhotskii-Weierstrass theorem was the first result characterizing the cluster set of an analytic function ff at an essential singularity. A stronger theorem from which the Casorati-Sokhotskii-Weierstrass theorem can be inferred is the Picard theorem.

14.2.2 Meromorphic Function

Meromorphic Function of one complex variable in a domain $\Omega \subset \mathbf{C}$ (or on a Riemann surface Ω)

A holomorphic function in a domain $\Omega \setminus \{\alpha_1, \alpha_2, \ldots\}$ which has at every singular point α_ν a pole (cf. Pole (of a function), i.e. α_ν is an isolated point of the set $\{\alpha_1, \alpha_2, \ldots\}$, which has no limit points in Ω, and $\lim_{z \to \alpha_\nu} |f(z)| = \infty$). The collection $M(\Omega)$ of all meromorphic functions in Ω is a field with respect to the usual pointwise operations followed by redefinition at the removable singularities.

The quotient ϕ / ψ of two arbitrary holomorphic functions in Ω, $\psi \not\equiv 0$, is a meromorphic function in Ω. Conversely, every meromorphic function in a domain $\Omega \subset \mathbf{C}$ (or on a non-compact Riemann surface Ω) can be expressed as $f = \phi / \psi$, $\psi \not\equiv 0$, where ϕ, ψ are holomorphic and have no common zeros in Ω. It follows that on a non-compact Riemann surface Ω the field $M(\Omega)$ coincides with the field of fractions of the ring $O(\Omega)$ of holomorphic functions in Ω.

Every meromorphic function $f \in M(\Omega)$ defines a continuous mapping $\tilde{f}$ of the domain Ω into the Riemann sphere $\mathbf{C} \cup \{\infty\}$, which is a holomorphic mapping relative to the standard complex structure on $\mathbf{C} \cup \{\infty\} \approx \mathbf{C}P^1$. Conversely, every holomorphic mapping $\tilde{f}: \Omega \to \mathbf{C} \cup \{\infty\}$, $\tilde{f} \not\equiv \infty$, defines a meromorphic function f in Ω: The set of poles of f coincides with the discrete set $\tilde{f}^{-1}(\infty)$ and $f(z)=\tilde{f}(z)\in\mathbf{C}$ if $z\in\Omega\setminus\tilde{f}^{-1}(\infty)$. Thus, the meromorphic functions of one variable may be identified with the holomorphic mappings ($\not\equiv\infty$) into the Riemann sphere.

The basic problems in the theory of meromorphic functions are those concerning the existence (and construction) of meromorphic functions with prescribed singularities.

I) One is given a (closed) discrete subset $\{a_1, a_2, \dots\}\subset\Omega$ and, at each point a_ν, the principal part of a Laurent expansion (cf. Laurent series)

$$f_\nu(z)=\sum_{j=1}^{n_\nu} c_{\nu j}(z-a_\nu)^{-j};$$

it is required to find a meromorphic function $f\in M(\Omega)$ with these principal parts, i.e. a holomorphic function f in $\Omega\setminus\{a_1, a_2, \dots\}$ such that $f-f_\nu$ is holomorphic in a neighbourhood of a_ν for each ν. If the number of points a_ν is finite, then (in a domain $\Omega\subset\mathbf{C}$) the problem is trivially solved by the function $\sum f_\nu$. In the general case this problem is solved by the Mittag-Leffler theorem: On every non-compact Riemann surface there exists a meromorphic function with given principal parts f_ν, $\nu=1,2,\dots$. On a compact Riemann surface (for instance, a torus) this problem has in general no solution — supplementary conditions concerning the compatibility of the principal parts must be imposed.

The second basic problem is conveniently formulated in the language of divisors (cf. Divisor), i.e. of mappings $D\colon\Omega\to\mathbf{Z}$ such that for every compactum $K\subset\Omega$ the number of points $z\in K$ at which $D(z)\neq 0$ is finite (the number $D(z)$ is called the multiplicity of D at z). Divisors can explicitly be written as formal sums $\sum k_\nu a_\nu$, where $a_\nu\in\Omega$ are the points at which $D(a_\nu)(=k_\nu)\neq 0$; in the case of finitely many terms the number $\sum k_\nu$ ($=\deg D$) is called the degree of the divisor D. For a meromorphic function f its divisor (f) is equal to zero everywhere apart from the zeros and poles of f, at which the multiplicity is set equal to the order of the zero or of the pole (poles have negative orders).

II) At the points of a (closed) discrete subset $\{a_1, a_2, \ldots\} \subset \Omega$ one is given "multiplicities" — integers $k_\nu \neq 0$. It is required to find a meromorphic function with zeros and poles of the respective multiplicities, i.e. a holomorphic function f in $\Omega \setminus \{a_1, a_2, \ldots\}$ such that $f(z)(z-a_\nu)^{-k_\nu}$ is holomorphic and does not vanish in a neighbourhood of the point a_ν, $\nu = 1, 2, \ldots$. In the case of finitely many points a_ν (and $\Omega \subset \mathbf{C}$) such a function is, for example, $f(z) = \prod (z-a_\nu)^{k_\nu}$. In the general case the problem is solved by Weierstrass' theorem: On a non-compact Riemann surface Ω, for every given divisor D there is a meromorphic function f with divisor (f) equal to D. For a compact Riemann surface Ω the holomorphic mapping into the Riemann sphere defined by a non-constant meromorphic function f is a branched covering, and hence the function f takes every value the same number of times; in particular, the number of zeros of f equals the number of its poles (multiplicities taken into account). Therefore, the condition $\deg (f) = 0$ is necessary in order that problem II admits a solution on a compact Riemann surface. In general, it is not sufficient; a necessary and sufficient condition for the existence of a meromorphic function with a given divisor is given by Abel's theorem .

Let D be a divisor on a compact Riemann surface Ω. The functions $f \in M(\Omega)$ satisfying the condition $(f) + D \geq 0$ form a finite-dimensional linear space O_D (over $\mathbf{C}$); if $\deg D < 0$, then $O_D = \{0\}$.

The Riemann–Roch theorem asserts that

$$\dim O_D - \dim O_{K-D} = \deg D - g + 1,$$

where K and g are the so-called canonical divisor and, respectively, the genus of the Riemann surface Ω. From this relation one can obtain many existence theorems (if $\dim O_{K-D} + \deg D > g$, then $\dim O_D \geq 2$, and hence O_D contains non-constant meromorphic functions). For example, on every compact Riemann surface Ω of genus g there is a meromorphic function which realizes a branched covering $f\colon \Omega \to \mathbf{C}\cup\{\infty\}$ with at most $g+1$ sheets.

An important place in the theory of meromorphic functions of one complex variable is occupied by value-distribution theory (Nevanlinna theory), which studies the distribution of the roots of the equations $f(z)=a$, $a\in\mathbf{C}\cup\{\infty\}$, when approaching the boundary of the domain

14.2.3 Meromorphic Functions of Several Complex Variables

Let Ω be a domain in $\mathbf{C}^n$ (or an n-dimensional complex manifold) and let $P\subset\Omega$ be a (complex-) analytic subset of codimension one (or empty). A holomorphic function f defined on $\Omega\setminus P$ is called a meromorphic function in Ω if for every point $p\in P$ one can find an arbitrarily small neighbourhood U of p in Ω and functions ϕ, ψ holomorphic in U without common non-invertible factors in $O(U)$, such that $f=\phi/\psi$ in $U\setminus P$. The set $P=P_f$ is called the polar set of the meromorphic function f. Its subset $N=N_f$, defined locally by the condition $\phi=\psi=0$, is called the set of (points of) indeterminacy of f; N is an analytic subset of Ω of (complex) codimension ≥ 2. At each point $p\in N$ the function f is essentially undefined: The limiting values of $f(z)$ for $z\to p$, $z\in\Omega\setminus P$, fill up the Riemann sphere $\mathbf{C}\cup\{\infty\}$. On the other hand, at the points of $P\setminus N$ the limit $\lim_{z\to p} f(z)=\infty$ exists, and upon redefining $f(p)=\infty$ if $p\in P\setminus N$, one obtains a holomorphic mapping of $\Omega\setminus N$ into the Riemann sphere. Conversely, if N is an arbitrary (possibly empty) complex-analytic subset of Ω of codimension ≥ 2, then every holomorphic mapping $f\colon \Omega\setminus N\to\mathbf{C}\cup\{\infty\}$ defines a meromorphic function on Ω that is equal to f on $\Omega\setminus P$, where $P=N\cup f^{-1}(\infty)$ is either an analytic subset of Ω of codimension 1 or is empty. Thus, a meromorphic function f in Ω can be defined as a holomorphic mapping into the Riemann sphere defined in the complement of an analytic subset $N_f \subset \Omega$ of codimension ≥ 2.

A third, completely localized, definition of meromorphic functions (equivalent to the one given above) is stated in the language of sheaves. Let O be the sheaf of germs of holomorphic functions on Ω, and for each point $z \in \Omega$ let M_z denote the field of fractions of the ring O_z (the stalk of the sheaf O over z). Then $M = \cup M_z$ is naturally endowed with the structure of a sheaf of fields, called the sheaf of germs of meromorphic functions in Ω. A meromorphic function in Ω is defined as a global section of M, i.e. a continuous mapping $f : z \to f_z$ such that $f_z \in M_z$ for all $z \in \Omega$. The sets P_f and N_f are defined as follows: If $f_z = \phi_z \,/\, \psi_z$, ϕ_z , $\psi_z \in O_z$, $\psi_z \neq 0$, then one may assume that ϕ_z and ψ_z are mutually prime, i.e. they have no common non-invertible factors in O_z; then $z \in P_f$ if $\psi_z\,(z) = 0$, while $z \in N$ if $\psi_z\,(z) = \phi_z\,(z) = 0$. The value at a point $z \notin P$ of the meromorphic function f thus defined is $\phi_z\,(z) \,/\, \psi_z\,(z)$.

As in the one-dimensional case, the collection of all meromorphic functions in Ω forms a field $M(\Omega)$ with respect to the pointwise algebraic operations with a subsequent redefinition at the removable singularities.

The closure $Z = Z_f$ of the zero set of a meromorphic function f, i.e. of the set $\{z \in \Omega \setminus P_f \,:\, f(z) = 0\}$, is an analytic subset of Ω of codimension one (or empty); the set of indeterminacy is $N = Z \cap P$. On Z_f and P_f one can define the order (multiplicity) of the zeros (or poles) of the meromorphic function f. If p is a regular point of the analytic set $Z \cup P$, then in some neighbourhood U of p the set $(Z \cup P) \cap U$ is connected and is given by an equation $g = 0$, $g \in O(U)$, where $dg \neq 0$ throughout U. Hence there is a maximal integer $k(p)$ such that the function $fg^{-k}(p)$ admits a holomorphic extension to U; this number is called the order (of the zero p if $p \in Z$, and of the pole p if $p \in N$) of the meromorphic function f at the point p. The function $k(p)$ is locally constant on the set of regular points of $Z \cup P$. Therefore one can attach to each meromorphic function in Ω its divisor $(f) = \sum k_\nu A_\nu$, where A_ν are the irreducible components of $Z_f \cup P_f$ and k_ν is the multiplicity (order) of f at the regular points of $Z \cup P$ that belong to A_ν (alternative notations: $(f) = D_f = \Delta_f$, etc.). On a compact complex manifold a meromorphic function is uniquely defined by its divisor, up to a multiplicative constant.

The problems solved in the one-dimensional case by the Mittag-Leffler and Weierstrass theorems are known in the higher-dimensional case as the first (additive) and the second (multiplicative) Cousin problems. Due to the complicated structure of the polar set P_f, the notion of a principal part of a meromorphic function is not defined in general, and accordingly the Cousin problems are formulated as follows.

I) Suppose that an open covering $\{U_\nu\}$ of the manifold Ω and in each U_ν a meromorphic function f_ν are given; it is required to find a meromorphic function $f \in M(\Omega)$ such that $f - f_\nu \in O(U_\nu)$ for all ν.

II) For a given divisor $D = \sum k_\nu A_\nu$ on Ω, find a meromorphic function $f \in M(\Omega)$ such that $(f) = D$.

The conditions of solvability of these problems in the higher-dimensional case are considerably more stringent than in the one-dimensional case.

The problem of representing a meromorphic function as a quotient of two holomorphic functions is called the Poincaré problem. The strong Poincaré problem is to represent a meromorphic function as a quotient of holomorphic functions the germs of which at each point $z \in \Omega$ are mutually prime in O_z. The Poincaré problem is unsolvable on a compact connected complex manifold if there are non-constant meromorphic functions on it. However, this problem is solvable in every domain $\Omega \subset \mathbf{C}^n$ and, in fact, in an arbitrary domain on a Stein manifold . The solvability of the strong Poincaré problem follows from that of the Cousin II problem (the converse is not true).

Functions $f_1,\dots, f_k \in M(\Omega)$ are said to be algebraically dependent if there is a polynomial $F \not\equiv 0$ in k variables with complex coefficients such that $F(f_1(z),\dots, f_k(z)) \equiv 0$ in the common domain of definition of the functions f_ν. The maximal number of algebraically-independent meromorphic functions on Ω is called the transcendence degree of the field $M(\Omega)$. On a compact complex manifold this number does not exceed the (complex) dimension of the manifold (Siegel's theorem); furthermore, the field $M(\Omega)$ has a finite number of generators.

On concrete complex manifolds, meromorphic functions may have supplementary properties. For instance, in the complex projective space $\mathbf{C}P^n$ the set of indeterminacy of any non-constant meromorphic function is not empty. Every meromorphic function on a projective algebraic variety is rational, i.e. is expressible as a quotient p/q of homogeneous polynomials in homogeneous coordinates. On algebraic varieties the field $M(\Omega)$ is quite rich. On the other hand, there exist complex manifolds (for example, some non-algebraic tori) on which every meromorphic function is constant. Higher-dimensional generalizations of the Riemann–Roch theorem are less effective, and existence theorems for various classes of meromorphic functions can only be obtained for some classes of complex manifolds.

14.2.4 Weierstrass Theorem

Infinite product theorem

Weierstrass' infinite product theorem [1]: For any given sequence of points in the complex plane $\mathbf{C}$,

$$0,\dots,0,\ \alpha_1,\ \alpha_2,\dots,$$

$$0 < |\alpha_k| \leq |\alpha_{k+1}|, \quad k = 1, 2, \dots; \quad \lim_{k\to\infty} |\alpha_k| = \infty,$$

there exists an entire function with zeros at the points α_k of this sequence and only at these points. This function may be constructed as a canonical product:

$$W(z) = z^{\lambda} \prod_{k=1}^{\infty} \left(1 - \frac{z}{\alpha_k}\right) e^{P_k(z)},$$

where λ is the multiplicity of zero in the sequence (1), and

$$P_k(z) = \frac{z}{\alpha_k} + \frac{z^2}{2\,\alpha_k^2} + \ldots + \frac{z^{m_k}}{2\,\alpha_k^{m_k}}\ .$$

The multipliers

$$W\left(\frac{z}{\alpha_k}; m_k\right) = \left(1 - \frac{z}{\alpha_k}\right) e^{P_k(z)}$$

are called Weierstrass prime multipliers or elementary factors. The exponents m_k are chosen so as to ensure the convergence of the product (2); for instance, the choice $m_k = k$ ensures the convergence of (2) for any sequence of the form (1).

It also follows from this theorem that any entire function $f(z)$ with zeros (1) has the form

$$f(z) = e^{g}(z) W(z),$$

where $W(z)$ is the canonical product (2) and $g(z)$ is an entire function (see also Hadamard theorem on entire functions).

Weierstrass' infinite product theorem can be generalized to the case of an arbitrary domain $D \subset \mathbf{C}$: Whatever a sequence of points $\{\alpha_k\} \subset D$ without limit points in D, there exists a holomorphic function f in D with zeros at the points α_k and only at these points.

The part of the theorem concerning the existence of an entire function with arbitrarily specified zeros may be generalized to functions of several complex variables as follows: Let each point α of the complex space $\mathbf{C}^n$, $n \geq 1$, be brought into correspondence with one of its neighbourhoods U_α and with a function f_α which is holomorphic in U_α. Moreover, suppose this is done in such a way that if the intersection $U_\alpha \cap U_\beta$ of the neighbourhoods of the points $\alpha, \beta \in \mathbf{C}^n$ is non-empty, then the fraction $f_\alpha / f_\beta \neq 0$ is a holomorphic function in $U_\alpha \cap U_\beta$. Under these conditions there exists an entire function f in $\mathbf{C}^n$ such that the fraction f / f_α is a holomorphic function at every point $\alpha \in \mathbf{C}^n$. This theorem is known as Cousin's second theorem

14.2.5 Riemann-Roch Theorem

A theorem expressing the Euler characteristic $\chi(\mathcal{E})$ of a locally free sheaf $\mathcal{E}$ on an algebraic or analytic variety X in terms of the characteristic Chern classes of $\mathcal{E}$ and X (cf. Chern class). It can be used to calculate the dimension of the space of sections of $\mathcal{E}$ (the Riemann–Roch problem).

The classical Riemann–Roch theorem relates to the case of non-singular algebraic curves X and states that for any divisor D on X,

$$l(D) - l(K_X - D) = \deg D - g + 1,$$

where $l(D) = \dim H^0(X, \mathcal{O}_X(D))$ is the dimension of the space of functions $f \in k(x)$ for which $(f) + D \geq 0$, K_X is the canonical divisor and g is the genus of X. In the middle of the 19th century B. Riemann used analytic methods to obtain the inequality

$$l(D) \geq \deg D - g + 1 .$$

The equality (1) was proved by E. Roch.

The Riemann–Roch theorem for curves is the one-dimensional case of the more general Riemann–Roch–Hirzebruch–Grothendieck theorem. Let X be a non-singular projective variety of dimension n, and let $H^\cdot X$ be an appropriate cohomology theory: either $H^\cdot X = H^\cdot(X, \mathbf{Q})$ are singular cohomology spaces when the basic field $k = \mathbf{C}$, or $H^\cdot X = A(X) \otimes \mathbf{Q}$ where $A(x)$ is a Chow ring, or $H^\cdot X$ is the ring associated to the Grothendieck ring $K^0(X)$ (see [2], [7]). Let $\mathcal{E}$ be a locally free sheaf of rank r on X. Universal polynomials for $\mathcal{E}$ with rational coefficients, $\mathrm{ch}\,(-)$ and $\mathrm{td}\,(-)$, in the Chern classes $c_i(\mathcal{E}) \in H^\cdot X$ of $\mathcal{E}$ are defined in the following way. The factorization

$$c_t(E) = c_0(\mathcal{E}) + \ldots + c_r(\mathcal{E})t^r = \prod_{i=1}^{r}(1 + a_i t)$$

is examined for the Chern polynomial, where the a_i are formal symbols. The exponential Chern character is defined by the formula

$$\mathrm{ch}\,(\mathcal{E}) = \sum_{i=1}^{r} e^{a_i} \quad \left(e^x = 1 + x + \frac{1}{2!}x^2 + \ldots\right),$$

and, accordingly, the Todd class is defined as

$$\mathrm{td}\,(\mathcal{E}) = \prod_{i=1}^{r} \frac{a_i}{1 - e^{-a_i}};$$

$\mathrm{ch}\,(\mathcal{E})$ and $\mathrm{td}\,(\mathcal{E})$ are symmetric functions in the a_i and they can be written as polynomials in $c_i(\mathcal{E})$.

The Riemann–Roch–Hirzebruch theorem: If X is a non-singular projective variety or a compact complex variety of dimension n and if $\mathcal{E}$ is a vector bundle of rank r on X, then

$$\chi\,(\mathcal{E}) = \deg\,(\,\mathrm{ch}\,(\mathcal{E})\,\mathrm{td}\,(\mathcal{T}_X))_n,$$

where $\mathcal{T}_X$ is the tangent sheaf on X and $\deg\,(.)_n$ denotes the component of degree n in $H^\cdot X$. This theorem was proved by F. Hirzebruch in the case of the ground field $\mathbf{C}$. When $n = 2$ and the invertible sheaf $\mathcal{E} = \mathcal{O}_X(D)$, it leads to the equation

$$\chi\,(\mathcal{O}_X(D)) = \frac{1}{2}D(D - K_X) + \frac{1}{12}(K_X^2 + c_2),$$

where $c_2 = c_2(X)$ is the second Chern class of the surface X and K_X is its canonical class. In particular, when $D = 0$ Noether's formula is obtained:

$$\chi\,(\mathcal{O}_X) = 1 + p_a = \frac{1}{12}(K_X^2 + c_2)\,.$$

For three-dimensional varieties $(n = 3)$ the theorem leads to

$$\chi(\mathcal{O}_X(D)) = \frac{1}{6}D^3 - \frac{1}{4}D^2 K_X + \frac{1}{12}D(K_X^2 + c_2) - \frac{1}{24}K_X c_2 .$$

In particular, when $D = 0$,

$$\chi(\mathcal{O}_X) = -\frac{1}{24}K_X c_2 .$$

In 1957, A. Grothendieck generalized the Riemann–Roch–Hirzebruch theorem to the case of a morphism of non-singular varieties over an arbitrary algebraically closed field (see [1]). Let $K_0 X$ and $K^0 X$ be the Grothendieck groups of the coherent and locally free sheafs on X, respectively (cf. Grothendieck group). The functor $K_0 X$ is a covariant functor from the category of schemes and proper morphisms into the category of Abelian groups. In this case, for a proper morphism $f: X \to Y$ the morphism $f: K_0 X \to K_0 Y$ is defined by the formula

$$f_!(\mathcal{F}) = \sum (-1)^i R^i f_*(\mathcal{F}),$$

where $\mathcal{F}$ is an arbitrary coherent sheaf on X and $K^0 X$ is a covariant functor into the category of rings. For regular schemes with an ample sheaf, the groups $K_0 X$ and $K^0 X$ coincide and are denoted by $K(X)$. The Chern character $\mathrm{ch} : K(X) \to H^\cdot X$ is a homomorphism of rings; $H^\cdot X$ is also a covariant functor: The Gizin homomorphism $f_* : H^\cdot X \to H^\cdot Y$ is defined. When $H^\cdot X = H^\cdot (X, \mathbf{Q})$, the homomorphism f_* is obtained from f_* for homology spaces using Poincaré duality. The theorem as generalized by Grothendieck expresses the measure of deviation from commutativity of the homomorphisms $f_!$ and ch.

The Riemann–Roch–Hirzebruch–Grothendieck theorem:
Let $f: X \to Y$ be a smooth projective morphism of non-singular projective varieties; then for any $x \in K(X)$ the equation

$$\mathrm{ch}\,(f_!(x)) = f_*(\,\mathrm{ch}\,(x)\,\mathrm{td}\,(\mathcal{T}_f\,))$$

in $H^\cdot X$ is true, where $\mathcal{T}_f = \mathcal{T}_X - f^*(\mathcal{T}_Y) \in K_X$ (the relative tangent sheaf of the morphism f).

When Y is a point, this theorem reduces to the Riemann–Roch–Hirzebruch theorem. There are generalizations (see [5], [6], [7]) when Y is a Noetherian scheme with an ample invertible sheaf, when f is a projective morphism whose fibres are locally complete intersections, and also to the case of singular quasi-projective varieties.

Several versions of the Riemann–Roch theorem are closely connected with the index problem for elliptic operators (see Index formulas). For example, the Riemann–Roch–Hirzebruch theorem for compact complex varieties is a particular case of the Atiyah–Singer index theorem.

Check in Progress-I

Note : Please give solution of questions in space give below:

Q. 1 State Picard's Theorem.

Solution :

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Q. 2 State Weierstrass' infinite product theorem .

Solution :

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14.3 CAUCHY-RIEMANN EQUATIONS

Theorem 3.1 (Cauchy-Riemann Equations). Suppose that

(3-1) $f(z) = f(x+iy) = u(x,y) + iv(x,y)$,

is differentiable at the point $z_0 = x_0 + iy_0$. Then the partial derivatives of u and v exist at the point (x_0, y_0),

and can be used to calculate the derivative at (x_0, y_0). That is,

(3-2) $f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)$,

and also

(3-3) $f'(z_0) = v_y(x_0, y_0) - i u_y(x_0, y_0)$.

Equating the real and imaginary parts of Equations (3-2) and (3-3) gives the so-called Cauchy-Riemann Equations:

(3-4) $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$.

Example 3.2. We know that $f(z) = z^2$ is differentiable and that $f'(z) = 2z$.

Furthermore, the Cartesian coordinate form for $f(z)$ is

$f(z) = z^2 = (x + i y)^2 = (x^2 - y^2) + i(2xy)$.

Use the Cartesian coordinate form of the Cauchy-Riemann equations and find $f'(z)$.

Solution. It is easy to verify that Cauchy-Riemann equations (3-4) are indeed satisfied:

$u_x(x, y) = 2x = v_y(x, y)$ and $u_y(x, y) = -2y = -v_x(x, y)$.

Using Equations (3-2) and (3-3), respectively, to compute $f'(z)$ gives

$f'(z) = u_x(x, y) + i v_x(x, y) = 2x + i 2y = 2z$,

and

$f'(z) = v_y(x, y) - i u_y(x, y) = 2x - i(-2y) = 2z$,

as expected.

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$u_x(x, y) = 2x = v_y(x, y)$ and $u_y(x, y) = -2y = -v_x(x, y)$.

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and

$f'(z) = v_y(x, y) - i u_y(x, y) = 2x - i(-2y) = 2z$,

as expected. **We are done.**

Example 3.3. Show that $f(z) = \overline{z}$ is nowhere differentiable.

Solution. We have $f(z) = f(x + i y) = x - i y = u(x, y) + i v(x, y)$, where

$u(x, y) = x$ and $v(x, y) = -y$.

Thus, for any point $z = x + i y$,

$u_x(x, y) = 1$ and $v_y(x, y) = -1$.

The Cauchy-Riemann equations (3-4) are not satisfied at any point $z = x + i y$, so we conclude that

$f(z) = \overline{z}$ is nowhere differentiable

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$f(z) = \overline{z}$ is nowhere differentiable.

Theorem 3.4 (Cauchy-Riemann conditions for differentiability). Assume that $f(z) = f(x + i y) = u(x, y) + i v(x, y)$ is a continuous function that is defined in some neighborhood of the point $z_0 = x_0 + i y_0$. If all the partial derivatives $u_x(x, y)$, $u_y(x, y)$, $v_x(x, y)$ and $v_y(x, y)$ are continuous at the point (x_0, y_0) and if the Cauchy-Riemann equations

(3-2) $u_x(x, y) = v_y(x, y)$ and $u_y(x, y) = -v_x(x, y)$

hold at $z_0 = x_0 + i y_0$, then $f(z)$ is differentiable at z_0, and the derivative $f'(z_0)$

can be computed with either formula (3-2) or (3-3), i. e.

$f'(z_0) = f'(x_0 + i y_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)$, or

$$f'(z_0) = f'(x_0 + i y_0) = v_y(x_0, y_0) - i u_y(x_0, y_0)$$

Example 3.5. At the beginning of this section (Equation (3.1)) we defined the function

$$f(z) = u(x, y) + i v(x, y) = x^3 - 3xy^2 + i(3x^2 y - y^3).$$

Show that this function is differentiable for all z, and find its derivative.

Solution. We compute $u_x(x, y) = 3x^2 - 3y^2 = v_y(x, y)$ and $u_y(x, y) = -6xy = -v_x(x, y)$, so the Cauchy-Riemann Equations (3-5), are satisfied. Moreover, the partial derivatives $u_x(x, y)$, $u_y(x, y)$, $v_x(x, y)$ and $v_y(x, y)$ are continuous everywhere.

By Theorem 3.4, $f(z) = x^3 - 3xy^2 + i(3x^2 y - y^3)$ is differentiable everywhere, and, from Equation (3-2),

$$f'(z) = u_x(x, y) + i v_x(x, y)$$

$$= 3x^2 - 3y^2 + i6xy$$

$$= 3(x^2 - y^2 + 2ixy)$$

$$= 3(x + iy)^2$$

$$= 3z^2$$

Alternatively, from Equation (3-3),

$$f'(z) = v_y(x, y) - i\,u_y(x, y)$$

$$= 3x^2 - 3y^2 - i(-6xy)$$

$$= 3(x^2 - y^2 + 2ixy)$$

$$= 3(x + iy)^2$$

$$= 3z^2$$

This result isn't surprising

because $(x + iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3)$,

and so the function $f(z)$ is really our old friend $f(z) = z^3$.

Solution. We

compute $u_x(x, y) = 3x^2 - 3y^2 = v_y(x, y)$ and

$u_y(x, y) = -6xy = -v_x(x, y)$, so the

Cauchy-Riemann equations , are satisfied. Moreover, the partial

derivatives

$u_x(x, y)$, $u_y(x, y)$, $v_x(x, y)$ and $v_y(x, y)$ are continuous

everywhere.

By Theorem 3.4, $f(z) = x^3 - 3xy^2 + i(3x^2y - y^3)$ is differentiable

everywhere, and, from Equation (3-3),

$$f'(z) = u_x(x, y) + i\,v_x(x, y)$$

$$= 3x^2 - 3y^2 + i\,6xy$$

$$= 3(x^2 - y^2 + 2ixy)$$

$$= 3(x + iy)^2$$

$$= 3z^2$$

Notes

Alternatively, from Equation (3-4),

$$f'(z) = v_y(x, y) - i\, u_y(x, y)$$

$$= 3x^2 - 3y^2 - i(-6xy)$$

$$= 3(x^2 - y^2 + 2ixy)$$

$$= 3(x + iy)^2$$

$$= 3z^2$$

This result isn't surprising

because $(x + iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3)$,

and so the function $f(z)$ is really our old friend $f(z) = z^3$.

We are done.

Exercise 1. Let $f(z) = e^x \cos y + i e^x \sin y$.

Show that both f(z) and f'(z) are differentiable for all z.

Answer. $f'(z) = e^x \cos y + i e^x \sin y$ and

$f''(z) = e^x \cos y + i e^x \sin y$ by Theorem 3.4.

Solution. $f(z) = f(x + iy) = e^x \cos y + i e^x \sin y$, and

$u(x, y) = e^x \cos y$, $v(x, y) = e^x \sin y$ so that

$u_x(x, y) = e^x \cos y$, $u_y(x, y) = -e^x \sin y$, $v_x(x, y) = e^x \sin y$,

$v_y(x, y) = e^x \cos y$.

The Cauchy Riemann equations are

$e^x \cos y = u_x(x, y) = v_y(x, y) = e^x \cos y$,

$-e^x \sin y = u_y(x, y) = -v_x(x, y) = -(e^x \sin y)$,

which hold for all z.

The partials are continuous everywhere, so

$f'(z) = u_x(x, y) + i v_x(x, y) = e^x \cos y + i e^x \sin y$,

for all z.

Exercise 2. A vector field $F(z) = U(x, y) + i V(x, y)$ is said to be irrotational if $U_y(x, y) = V_x(x, y)$.

It is said to be solenoidal if $U_x(x, y) = -V_y(x, y)$.

If f(z) is an analytic function, show that $F(z) = \overline{f(z)}$ is both irrotational and solenoidal.

Solution. Let $f(z) = u(x, y) + i v(x, y)$ be an analytic function.

By definition, $F(z) = \overline{f(z)} = u(x, y) - i v(x, y)$, so

$u_x(x, y) = U_x(x, y)$, $u_y(x, y) = U_y(x, y)$, $v_x(x, y) = -V_x(x, y)$, and $v_y(x, y) = -V_y(x, y)$.

By the Cauchy-Riemann equations,

$U_y(x, y) = u_y(x, y) = -v_x(x, y) = -(-V_x(x, y)) = V_x(x, y)$, and

$U_x(x, y) = u_x(x, y) = v_y(x, y) = -V_y(x, y)$.

Therefore $F(z) = \overline{f(z)}$ is both irrotational and solenoidal.

Exercise 3. $\int_0^\infty \frac{1}{x^3 + 1} \, dx$.

Hint. Use the contour $C = L_1 + C_R - L_2$ shown in the figure below,

and establish that $\int_{L_2} \frac{1}{z^3 + 1} \, dz = e^{i 2\pi/3} \int_0^\infty \frac{1}{x^3 + 1} \, dx$.

Answer. $\int_0^\infty \frac{1}{x^3 + 1} \, dx = \frac{2\sqrt{3}}{9} \pi$.

Solution. The complex integrand is $f(z) = \frac{1}{z^3 + 1}$.

The denominator is $z^3 + 1 = (z + 1)(z^2 - z + 1) =$

$(z + 1)\left(z - \frac{1}{2} - \frac{\sqrt{3}}{2} i\right)\left(z - \frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$.

The zeros of $z^3 + 1$ are $z_1 = -1$, $z_2 = \frac{1}{2} + \frac{\sqrt{3}}{2} i$, $z_2 = \frac{1}{2} - \frac{\sqrt{3}}{2} i$.

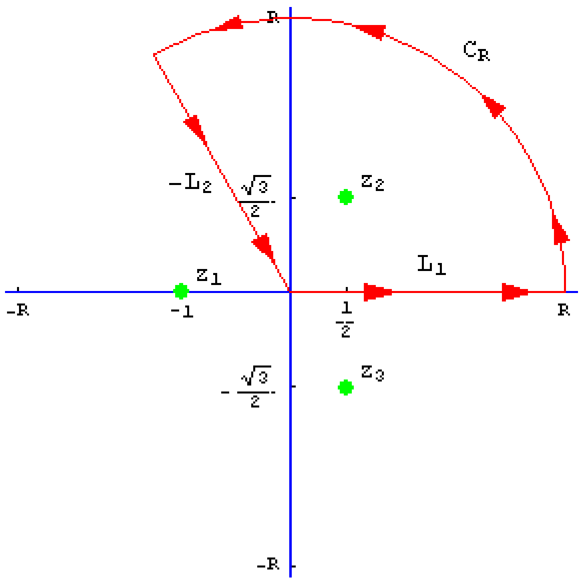
Notes

The poles of $f(z) = \frac{1}{z^3 + 1}$ are

$$z_1 = -1,\ z_2 = \frac{1}{2} + \frac{\sqrt{3}}{2} i,\ z_3 = \frac{1}{2} - \frac{\sqrt{3}}{2} i.$$

We will use the contour $C = L_1 + C_R - L_2$ consisting of the semi-circle C_R and the interval $L_1 = \{x : 0 \le x \le R\}$,

and the segment $L_2 = \left\{z = r e^{i \frac{2\pi}{3}} : 0 \le r \le R\right\}$. The pole $z_1 = \frac{1}{2} + \frac{\sqrt{3}}{2} i$ lies in inside $C = L_1 + C_R - L_2$.



The point $z_1 = \frac{1}{2} + \frac{\sqrt{3}}{2} i$ lies in inside $C = L_1 + C_R - L_2$.

Using (Cauchy's Residue Theorem), we obtain

$$\int_C f(z)\, dz = 2\pi i \operatorname{Res}[f, z_1]$$

$$= 2\pi i \operatorname{Res}\left[f, \frac{1}{2} + \frac{\sqrt{3}}{2} i\right]$$

$$= 2\pi i \left(-\frac{1}{6} - \frac{1}{2\sqrt{3}} i\right)$$

$$= \frac{\pi}{\sqrt{3}} - \frac{\pi}{3} i$$

Since $z_1 = \frac{1}{2} + \frac{\sqrt{3}}{2} i$ is a simple pole, by the residue is calculated as follows:

$$\text{Res}[f, z_1] = \lim_{z \to z_1} (z - z_1) f(z)$$

$$= \lim_{z \to \left(1+\sqrt{3}\, i\right)/2} \left(z - \frac{1}{2} - \frac{\sqrt{3}}{2}\, i\right) \frac{1}{z^3 + 1}$$

$$= \lim_{z \to \left(1+\sqrt{3}\, i\right)/2} \frac{z - \left(1 + \sqrt{3}\, i\right)/2}{z^3 + 1} = \text{'}\, \frac{0}{0}\, \text{'}$$

$$= \lim_{z \to \left(1+\sqrt{3}\, i\right)/2} \frac{1}{3 z^2} = \text{'}\, \frac{0}{0}\, \text{'}$$

$$= \frac{1}{3\left(\left(1 + \sqrt{3}\, i\right)/2\right)^2}$$

$$= \frac{1}{3\left(\left(-1 + \sqrt{3}\, i\right)/2\right)}$$

$$= -\frac{1}{6} - \frac{1}{2\sqrt{3}}\, i$$

On the segment L_2 we use the change of variable $z = r e^{i 2\pi/3}$ and $dz = e^{i 2\pi/3} dr$ for $0 \le r \le R$,

and we can calculate the contour integral over $C = L_1 + C_R - L_2$ in the following manner:

$$\int_C f(z)\, dz = \int_{L_1} f(z)\, dz + \int_{C_R} f(z)\, dz - \int_{L_2} f(z)\, dz$$

$$= \int_{L_1} f(z)\, dz + \int_{C_R} f(z)\, dz - \int_{L_2} f(z)\, dz$$

$$= \int_0^R \frac{1}{x^3 + 1}\, dx + \int_{C_R} f(z)\, dz - \int_0^R \frac{1}{\left(r e^{i 2\pi/3}\right)^3 + 1} e^{i 2\pi/3}\, dr$$

$$= \int_0^R \frac{1}{x^3 + 1}\, dx + \int_{C_R} f(z)\, dz - e^{i 2\pi/3} \int_0^R \frac{1}{\left(r e^{i 2\pi}\right)^3 + 1}\, dr$$

$$= \int_0^R \frac{1}{x^3 + 1}\, dx + \int_{C_R} f(z)\, dz - e^{i 2\pi/3} \int_0^R \frac{1}{x^3 + 1}\, dx$$

$$= \left(1 - e^{i 2\pi/3}\right) \int_0^R \frac{1}{x^3 + 1}\, dx + \int_{C_R} f(z)\, dz$$

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$$.

Now use the value contour integral that we obtained by the residue calculus

$$\frac{\pi}{\sqrt{3}} - \frac{\pi}{3} i = \int_C f(z)\,dz$$

$$= \lim_{R\to\infty} \int_C f(z)\,dz$$

$$= \lim_{R\to\infty} \left(1 - e^{i 2\pi/3}\right) \int_0^R \frac{1}{x^3+1}\,dx + \lim_{R\to\infty} \int_{C_R} f(z)\,dz$$

$$= \left(1 - e^{i 2\pi/3}\right) \int_0^\infty \frac{1}{x^3+1}\,dx + 0$$

$$= \left(\frac{3}{2} - \frac{\sqrt{3}}{2} i\right) \int_0^\infty \frac{1}{x^3+1}\,dx$$

An easy computation will now finish our work

$$\int_0^\infty \frac{1}{x^3+1}\,dx = \left(\frac{\pi}{\sqrt{3}} - \frac{\pi}{3} i\right) \Big/ \left(\frac{3}{2} - \frac{\sqrt{3}}{2} i\right)$$

$$= \frac{2}{3\sqrt{3}} \pi$$

$$= \frac{2\sqrt{3}}{9} \pi$$

We aredone.

14.4 SUMMARY

We study in this unitBorel fixed-Point Theorem and its proof. We study Borel function and its statement . We study Fourier-Borel transform and its proof. We study Casorati-Sokhotskii-Weierstrass Theorem.We study Meromorphic Function. We study Weierstrass Theorem and its statement with proof. We study Riemann-Roch Theorem. We study Cauchy-Riemann Theorem with examples and proof with definition.

14.5 KEYWORD

Borel Fixed-Point : If G is a connected, solvable, linear algebraic group acting regularly on a non-empty, complete algebraic variety V over an algebraically closed field k, then there is a G fixed-point of V. A more general version of the theorem holds over a field k that is not necessarily algebraically closed

Riemann-Roch : It relates the complex analysis of a connected compact *Riemann* surface with the surface's purely topological genus g, in a way that can be carried over into purely algebraic settings.

Meromorphic : a *meromorphic* function on an open subset D of the complex plane is a function that is holomorphic on all of D except for a set of isolated points, which are poles of the function

Irrotational : (especially of fluid motion) not rotational; having no rotation

14.6 QUESTIONS FOR REVIEW

Exercise 1. Let $f(z) = u(x, y) + i v(x, y)$ be a differentiable function.

Verify the identity $|f'(z)|^2 = (u_x(x, y))^2 + (v_x(x, y))^2$
$= (u_y(x, y))^2 + (v_y(x, y))^2$.

Exercise 2. Find the constants a and b such that
$f(x + i y) = 2x - y + i(ax + by)$ is differentiable for all z.

Exercise 3. Let f(z) be differentiable at the point $z_0 = r_0 e^{i\theta_0}$.

Let z approach z_0 along the ray $r > 0, \theta = \theta_0$ and how that Equation
$f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)$ holds.

Exercise 4. Let $f(z) = e^x \cos y + i e^x \sin y$.

Show that both f(z) and f'(z) are differentiable for all z.

Exercise 5. A vector field $F(z) = U(x, y) + i V(x, y)$ is said to be irrotational if $U_y(x, y) = V_x(x, y)$.

It is said to be solenoidal if $U_x(x, y) = -V_y(x, y)$.

If f(z) is an analytic function, show that $F(z) = \overline{f(z)}$ is both irrotational and solenoidal.

Exercise 6. Determine where the following functions are differentiable and where they are analytic. Explain!

$f(z) = f(x, y) = x^3 + 3xy^2 + i(y^3 + 3x^2 y)$.

14.7 SUGGESTION READING AND REFERENCES

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14.8 ANSWER TO CHECK YOUR PROGRESS

Check In Progress-I

Answer Q. 1 Check in Section 1.3

2 Check in Section 1.4

Check In Progress-II

Answer Q. 1 Check in section 2

2 Check in Section 2.4